

n -dimensional global correspondences of Langlands over singular schemes (II)

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“This paper is dedicated to R. Thom who, by his enthusiasm, convinced me of the importance of the singularities and, by his patience, backed me up along my long research towards the blowups of the versal deformations, the geometries of these processes and the (strange) attractors tied up to these”.

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Abstract

A rather complete phenomenology of the singularities is developed according to a new algebraic point of view in the frame of Langlands global correspondences.

That is to say, a process of:

- singularizations and versal deformations of these,
- singularizations and monodromies of these,

is envisaged on all the sections of sheaves of differentiable (bi)functions on (bi)linear algebraic (semi)groups constituting the n -dimensional representations of the global Weil groups.

To get the searched holomorphic and cuspidal representations, it is necessary to consider:

- the resolutions of singularities and the blowups of the versal deformations;
- the resolutions of the singularities in the monodromy cases.

Furthermore, the geometry of the versal deformations and of their blowups is studied, as well as the associated dynamics leading to the consideration of singular hyperbolic attractors and of singular strange attractors.

Introduction

This paper constitutes the second part of the n -dimensional global correspondences of Langlands [Lan]: it points out the cases in which these correspondences can still be stated while the sheaf of differentiable (bi)functions on a bilinear algebraic semigroup, constituting the n -dimensional representation of the global Weil group, is affected by all kinds of singularities together with deformations and blowups of these.

The phenomenology of the singularities can be split into two sets based on contracting or dilating morphisms characterized respectively by underlying topological subsets getting closer and closer or farther and farther.

In the set characterized by **dilating morphisms**, we find:

- a) **the desingularizations**, or the resolutions of the singularities, of a singular scheme consisting in monomializing polynomial ideals by sequences of blowups [Hau].
- b) **the monodromy** transformations of a singular scheme arising in an expanding phase in such a way that non-singular fibres can be generated.

while we have in the set characterized by **contracting morphisms**:

- a) **the singularizations** introduced as the inverse morphisms of the resolutions of singularities, and which are defined by sequences of contracting surjective morphisms producing singular loci.
- b) **the versal deformations** of these singularities which can be interpreted as extensions of the sequences of contracting surjective morphisms of singularizations, recovering then the classical definition of the versal deformation as (contracting) fibre bundles of which fibres are the bases of the versal deformations.
- c) **the blowups of the versal deformations**, introduced in [Pie3], [Pie4] as the inverse morphisms of the versal deformations: they are based upon Galois antiautomorphisms and are also called spreading-out isomorphisms.

The considered mathematical frame, recalled in chapter 1, is the same as the one which was envisaged in the first part [Pie1] of the n -dimensional global correspondences of Langlands, that is to say, considering:

- sets F_v^+ (resp. $F_{\bar{v}}^+$) of r packets of left (resp. right) real pseudo-ramified equivalent completions associated with the left (resp. right) (algebraically closed) extension (semi)field of a number field of characteristic zero and characterized by increasing Galois extension degrees being integers modulo N .

- sets F_ω (resp. $F_{\bar{\omega}}$) of r packets of left (resp. right) complex pseudo-ramified equivalent completions covered by their real equivalents.
- bilinear algebraic semigroups $\mathrm{GL}_n(F_{\bar{\omega}} \times F_\omega) \equiv T_n^t(F_{\bar{\omega}}) \times T_n(F_\omega)$ and $\mathrm{GL}_n(F_{\bar{v}}^+ \times F_v^+) \equiv T_n^t(F_{\bar{v}}^+) \times T_n(F_v^+)$ respectively over products of complex and real completions in such a way that their representation spaces $G^{(n)}(F_{\bar{\omega}} \times F_\omega)$ and $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ be n^2 -dimensional complex and real bilinear affine semigroups [Pie1] decomposing into sets $\{g_{R \times L}^{(n)}[j, m_j]\}_{j=1, m_j}^r$ and $\{g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]\}_{j_\delta=1, m_{j_\delta}}^r$ of r packets of complex and real equivalent conjugacy class representatives:

each conjugacy class representative $g_{R \times L}^{(n)}[j, m_j]$ is the product of a right n -dimensional complex algebraic semitorus $g_R^{(n)}[j, m_j]$ by its left equivalent $g_L^{(n)}[j, m_j]$ verifying $g_R^{(n)}[j, m_j] \simeq (F_{\bar{\omega}_{j, m_j}})^n$ and $g_L^{(n)}[j, m_j] \simeq (F_{\omega_{j, m_j}})^n$ where $F_{\bar{\omega}_{j, m_j}}$ (resp. $F_{\omega_{j, m_j}}$) is the (j, m_j) -th corresponding complex completion of $F_{\bar{\omega}}$ (resp. F_ω) and each conjugacy class representative $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ is the product of a right n -dimensional real algebraic semitorus $g_R^{(n)}[j_\delta, m_{j_\delta}]$ by its left equivalent $g_L^{(n)}[j_\delta, m_{j_\delta}]$ verifying $g_R^{(n)}[j_\delta, m_{j_\delta}] \simeq (F_{\bar{v}_{j_\delta, m_{j_\delta}}}^+)^n$ and $g_L^{(n)}[j_\delta, m_{j_\delta}] \simeq (F_{v_{j_\delta, m_{j_\delta}}}^+)^n$ where $F_{\bar{v}_{j_\delta, m_{j_\delta}}}^+$ (resp. $F_{v_{j_\delta, m_{j_\delta}}}^+$) is the (j_δ, m_{j_δ}) -th corresponding real completion of $F_{\bar{v}}^+$ (resp. F_v^+).

- a (bisemi)sheaf $\theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}}$ of differentiable bifunctions $\phi_{G_{j_R}^{(n)}}^{(n)}(x_{g_{j_\delta}}) \otimes \phi_{G_{j_L}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ on the conjugacy class representatives $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ of the real bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ and a bisemisheaf $\theta_{G_R^{(n)}}^{(\mathbb{C})} \times \theta_{G_L^{(n)}}^{(\mathbb{C})}$ of complex-valued differentiable bifunctions $\phi_{G_R^{(n)}}^{(\mathbb{C})}(x_{g_R}) \otimes \phi_{G_L^{(n)}}^{(\mathbb{C})}(x_{g_L})$ on the conjugacy class representatives $g_{R \times L}^{(n)}[j, m_j]$ of the complex bilinear algebraic semigroup $G^{(n)}(F_{\bar{\omega}} \times F_\omega)$.

The most important part of this paper, i.e. chapters 2, 3 and 4, concerns the study of:

- degenerate singularities on the sections $\phi_{G_{j_L}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}^{(n)}}^{(n)}(x_{g_{j_\delta}})$) of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) as resulting from contracting, generally surjective, morphisms.
- the existence of Langlands global correspondences in the “singular” context.

First of all, a **process of singularization** is introduced in chapter 2, as being the inverse of monoidal transformations. It consists in projecting a sequence of normal crossings divisors defined on irreducible completions of rank N onto a singular locus which then becomes the homotopic image of these normal crossings divisors under a sequence of contracting surjective morphisms.

For example, a singularization of type A_k , given by the germs $y_L = x_L^{k+1}$ (resp. $y_R = x_R^{k+1}$) of differentiable functions $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}})$) on the j_δ -th conjugacy class of $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_v^+)$), is generated by a sequence of $k + 1$ contracting surjective morphisms, being in fact fibre bundles whose contracting fibres are the homotopic images of the normal crossing divisors.

Outside of the singular locus, the contracting morphism of singularization is an isomorphism. Due to the relative small topological space on which is defined the semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$), it will be assumed that on every function $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}}) \in \theta_{G_L^{(n)}}$ (resp. cofunction $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}}) \in \theta_{G_R^{(n)}}$) a same kind of singularity (or set of singularities) is generated by the singularization process.

So, the singularization of $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) is a process transforming it into a singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) whose sections $\phi_{G_{j_\delta L}}^{*(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_\delta R}}^{*(n)}(x_{g_{j_\delta}})$) are the differentiable functions $\phi_{G_{j_\delta L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_\delta R}}^{(n)}(x_{g_{j_\delta}})$) endowed with germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) having (degenerate) singularities assumed to be of corank 1 (to simplify the handling).

The versal deformation $\theta_{G_L^{(n)}}^{\text{vers}} = \theta_{G_L^{(n)}}^* \times \theta_{S_L}$ (resp. $\theta_{G_R^{(n)}}^{\text{vers}} = \theta_{G_R^{(n)}}^* \times \theta_{S_R}$) of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$), whose sections are endowed with germs having degenerate singularities (of corank 1), can be interpreted as the total space of a fibre bundle D_{S_L} (resp. D_{S_R}) of which fibre θ_{S_L} (resp. θ_{S_R}) is the family of the (semi)sheaves of the base S_L (resp. S_R) of the versal deformation.

In this context, the versal deformation consists in an extension of the singularization process in the sense that it is generated by a sequence of contracting morphisms extending the sequence of contracting surjective morphisms of singularizations by projecting sets of normal crossing divisors in the neighbourhoods of the singular loci according to the finite determinacies of the considered degenerate singularities on the sections of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$). If we refer to the degenerate germ $y_L = x_L^{k+1}$ (resp. $y_R = x_R^{k+1}$) of type A_k , its versal deformation will then result from a sequence of $(k - 2)$ contracting morphisms extending the sequence of contracting surjective morphisms of singularization in $(k - 2)$ dimensions in such a way that a sequence of $(k - 2)$ (sets of) normal crossings divisors be projected in the neighbourhood of the singular locus.

This constitutes the content of chapter 2, section 2, while section 3 deals with the **geometry of the versal deformation**:

It is proved that:

- a) the geometry is hyperbolic in the neighbourhood of the singular locus of a not unfolded degenerate singular germ of corank $m \leq 3$ and multiplicity i , $1 \leq i \leq n$, in the sense that:
- the limit set of the Kleinian group acting in the neighbourhood of a singular locus corresponds precisely to this singular locus.
 - the ordinary set of the Kleinian group can be associated with the neighbourhood of the singular locus and is characterized by a hyperbolic metric.
- b) the neighbourhood of the unfolded germ on the section $\phi_{G_{j\delta_L}}^{(n)}(x_{g_{j\delta}})$ (resp. $\phi_{G_{j\delta_R}}^{(n)}(x_{g_{j\delta}})$) of the unfolded semisheaf $\theta_{G_L^{(n)}}^{\text{vers}}$ (resp. $\theta_{G_R^{(n)}}^{\text{vers}}$) is characterized by a spherical geometry except in the neighbourhood of the singular locus where the geometry is hyperbolic.

Chapter 3 envisages the blowup of the versal deformation as well as the study of the strange attractors related to the versal deformations of singular germs.

The blowup of the versal deformation

$$\theta_{G_L^{(n)}}^{\text{vers}} = \theta_{G_L^{(n)}}^* \times \theta_{S_L} \quad (\text{resp.} \quad \theta_{G_R^{(n)}}^{\text{vers}} = \theta_{G_R^{(n)}}^* \times \theta_{S_R})$$

of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) consists essentially in an algebraic endomorphism Π_{S_L} (resp. Π_{S_R}), based on Galois antiautomorphisms, pulling out partially or completely the sheaves $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$), $1 \leq i \leq s$, of the fibre (corank 1 case)

$$\begin{aligned} \theta_{S_L} &= \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_L^i), \dots, \theta^1(\omega_L^s)\} \\ (\text{resp.} \quad \theta_{S_R} &= \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}) \end{aligned}$$

of the versal deformation from the $(n-1)$ -dimensional coefficient sheaf

$$\begin{aligned} \theta_L(a) &= \{\theta_L^{n-1}(a_1), \dots, \theta_L^{n-1}(a_i), \dots, \theta_L^{n-1}(a_s)\} \in \theta_{G_L^{(n)}} \\ (\text{resp.} \quad \theta_R(a) &= \{\theta_R^{n-1}(a_1), \dots, \theta_R^{n-1}(a_i), \dots, \theta_R^{n-1}(a_s)\} \in \theta_{G_R^{(n)}}) \end{aligned}$$

on which θ_{S_L} (resp. θ_{S_R}) was projected.

This blowup is maximal when all the base (semi)sheaves of θ_{S_L} (resp. θ_{S_R}) have been pulled out from $\theta_L(a)$ (resp. $\theta_R(a)$).

The blowup is complete if it is given by the composition of maps

$$(S \circ T)_L = (\tau_{V_{\omega_L}} \circ \Pi_{S_L}) \quad (\text{resp.} \quad (S \circ T)_R = (\tau_{V_{\omega_R}} \circ \Pi_{S_R}))$$

where $\tau_{V_{\omega_L}}$ (resp. $\tau_{V_{\omega_R}}$) is the projective map of the tangent bundle projecting all the disconnected base (semi)sheaves $\theta_L^1(\omega_L^i)$ (resp. $\theta_R^1(\omega_R^i)$) of θ_{S_L} (resp. θ_{S_R}) in the vertical

tangent spaces: this blowup then constitutes an extension of the quotient algebra of the versal deformation of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$). When it is maximal, it is given by the map:

$$\begin{aligned} (S \circ T)_L^{\max} : \theta_{G_L^{(n)}}^* \times \theta_{S_L} &\longrightarrow \theta_{G_L^{(n)}}^* \cup \theta_{S_L} \\ (\text{resp. } (S \circ T)_R^{\max} : \theta_{G_R^{(n)}}^* \times \theta_{S_R} &\longrightarrow \theta_{G_R^{(n)}}^* \cup \theta_{S_R}) \end{aligned}$$

and corresponds to the inverse of the versal deformation D_{S_L} (resp. D_{S_R}) according to:

$$(S \circ T)_L^{\max} = (D_{S_L})^{-1} \quad (\text{resp. } (S \circ T)_R^{\max} = (D_{S_R})^{-1}).$$

The family of disconnected base semisheaves are then glued together and cover partially the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$): they are labelled $\theta_{(S \circ T)(1)_L}^*$ (resp. $\theta_{(S \circ T)(1)_R}^*$) and verify $\theta_{(S \circ T)(1)_L}^* \simeq \theta_{S_L}$ (resp. $\theta_{(S \circ T)(1)_R}^* \simeq \theta_{S_R}$). But, $\theta_{(S \circ T)(1)_L}^*$ (resp. $\theta_{(S \circ T)(1)_R}^*$) can be affected by singularities on its sections involving versal deformations and blowups.

Section 3.2 envisages the versal deformation and its blowup from a differentiable and dynamical point of view.

The dynamics is envisaged around singularities on the sections of the tangent bundle on the conjugacy class representatives of the algebraic semigroup $G_L^{(n)}(F_v^+) \simeq T_n(F_v^+)$ (resp. $G_R^{(n)}(F_v^+) \simeq T_n(F_v^+)$). Then, the neighbourhood of the singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) on the n -dimensional real-valued differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$) of the space of sections $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) of the tangent bundle on $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) is a **singular hyperbolic attractor** Λ_L^{TAN} (resp. Λ_R^{TAN}) with respect to the diffeomorphisms $\text{Diff}_L(T(G_L^{(n)}(F_v^+)))$ (resp. $\text{Diff}_R(T(G_R^{(n)}(F_v^+)))$).

And, the versal unfolding of the germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) involves the map:

$$\begin{aligned} \vee D_{\Lambda_L} : \Lambda_L^{\text{TAN}} &\longrightarrow \Lambda_{\text{str}_L}^{\text{TAN}} \\ (\text{resp. } \vee D_{\Lambda_R} : \Lambda_R^{\text{TAN}} &\longrightarrow \Lambda_{\text{str}_R}^{\text{TAN}}) \end{aligned}$$

of the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) into the **singular strange attractor**

$$\begin{aligned} \Lambda_{\text{str}_L}^{\text{TAN}} &= \Lambda_L^{\text{TAN}} \times \Lambda_{\text{unf}_L}^{\text{TAN}} \\ (\text{resp. } \Lambda_{\text{str}_R}^{\text{TAN}} &= \Lambda_R^{\text{TAN}} \times \Lambda_{\text{unf}_R}^{\text{TAN}}) \end{aligned}$$

where $\Lambda_{\text{unf}_L}^{\text{TAN}}$ (resp. $\Lambda_{\text{unf}_R}^{\text{TAN}}$) is an unfolded attractor which can be expressed according to:

$$\Lambda_{\text{unf}_L}^{\text{TAN}} = \cup \Lambda_{\omega_{j_\delta L}^i}^{\text{TAN}} \quad (\text{resp. } \Lambda_{\text{unf}_R}^{\text{TAN}} = \cup \Lambda_{\omega_{j_\delta R}^i}^{\text{TAN}})$$

with $\Lambda_{\omega_{j\delta_L}^i}^{\text{TAN}}$ (resp. $\Lambda_{\omega_{j\delta_R}^i}^{\text{TAN}}$) a singular hyperbolic attractor resulting from a singularity on the generator $\omega_{j\delta_L}^i$ (resp. $\omega_{j\delta_R}^i$) of the versal deformation of $\phi_{\delta_j}(\omega_L)$ (resp. $\phi_{\delta_j}(\omega_R)$).

Finally, a blowup of the singular strange attractor $\Lambda_{\text{str}_L}^{\text{TAN}}$ (resp. $\Lambda_{\text{str}_R}^{\text{TAN}}$) can disconnect the singular hyperbolic attractors $\Lambda_{\omega_{j\delta_L}^i}^{\text{TAN}}$ (resp. $\Lambda_{\omega_{j\delta_R}^i}^{\text{TAN}}$) from the basic singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_r^{TAN}).

In chapter 4, it is analysed in what extend it is possible to develop **global correspondences of Langlands** for a bisemisheaf of differentiable functions on the real algebraic bilinear semigroup $(G^{(n)}(F_v^+ \times F_v^+))$ affected by degenerate singularities.

Recall that a global correspondence consists in a bijection between the n -dimensional irreducible representation $\text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ of the product, right by left, of Weil groups and the irreducible cuspidal representation $\text{Irr ELLIP}(\text{GL}_n(\mathbb{A}_{F_v^{+,T}} \times \mathbb{A}_{F_v^{+,T}}))$ of $\text{GL}_n(F_v^+ \times F_v^+)$ as developed in [Piel].

Now, $\text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ is given by the bilinear affine semigroup $G^{(n)}(F_v^+ \times F_v^+)$ or by the semisheaf $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ on it.

But, **under singularization, versal deformation and blowup of it, the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ has been transformed into:**

$$\begin{aligned} \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} &\xrightarrow{\bar{\rho}_{G_R} \times \bar{\rho}_{G_L}} \theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^* \xrightarrow{D_{S_R} \times D_{S_L}} (\theta_{G_R^{(n)}}^* \times \theta_{S_R}) \otimes (\theta_{G_L^{(n)}}^* \times \theta_{S_L}) \\ &\xrightarrow{(S \circ T)_R^{\max} \times (S \circ T)_L^{\max}} (\theta_{G_R^{(n)}}^* \cup \theta_{(S \circ T)(1)_R}^*) \otimes (\theta_{G_L^{(n)}}^* \cup \theta_{(S \circ T)(1)_L}^*) \end{aligned}$$

where

- $\bar{\rho}_{G_R} \times \bar{\rho}_{G_L}$ is the contracting morphism of singularization.
- $D_{S_R} \times D_{S_L}$ is the contracting morphism of versal deformation.
- $(S \circ T)_R^{\max} \times (S \circ T)_L^{\max}$ is the blowup of the versal deformation.

So, $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ has generated under $((S \circ T)_R^{\max} \circ D_{S_R} \circ \bar{\rho}_{G_R}) \times ((S \circ T)_L^{\max} \circ D_{S_L} \circ \bar{\rho}_{G_L})$ the singular bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and the singular compactified base bisemisheaf $(\theta_{(S \circ T)(1)_R}^* \otimes \theta_{(S \circ T)(1)_L}^*)$ of the blowup of the versal deformation.

But, these bisemisheaves $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and $(\theta_{(S \circ T)(1)_R}^* \otimes \theta_{(S \circ T)(1)_L}^*)$, affected by singularities, cannot be endowed with a cuspidal representation.

To reach this objective, it is necessary to:

- 1) desingularize those bisemisheaves.
- 2) submit them to a toroidal compactification.

The desingularization corresponds to the classical monoidal transformations and is reached by a set of inverse morphisms of those defining a singularization as developed in section 2.1.

Before considering the cuspidal representations of these bisemisheaves $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and $(\theta_{(S \circ T)(1)_R}^* \otimes \theta_{(S \circ T)(1)_L}^*)$, we can at this stage envisage holomorphic representations of the corresponding desingularized bisemisheaves $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ and $(\theta_{(S \circ T)(1)_R} \otimes \theta_{(S \circ T)(1)_L})$. We shall briefly recall how to get a holomorphic representation for $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$, taking into account that the same procedure can be applied to $(\theta_{(S \circ T)(1)_R} \otimes \theta_{(S \circ T)(1)_L})$.

The **global holomorphic representation** $\text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ of the bisemisheaf $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ is given by the morphism:

$$\text{Irr hol}_{\theta_{G_R \times L}}^{(n)} : \quad \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} \longrightarrow f_{\overline{v}}(z^*) \otimes f_v(z)$$

where $f_{\overline{v}}(z^*) \otimes f_v(z)$ is the holomorphic bifunction (i.e. product of a holomorphic function by the corresponding symmetric cofunction) obtained by gluing together and adding the bisections of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$.

So, in a few words, a singular bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$, submitted to a versal deformation transforming it into $(\theta_{G_R^{(n)}}^* \times \theta_{S_R}) \otimes (\theta_{G_L^{(n)}}^* \times \theta_{S_L})$, can be endowed with a holomorphic representation if a blowup of the versal deformation is considered as well as a desingularization of the resulting singular bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$.

To get a **cuspidal representation of the desingularized bisemisheaves** $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ and $(\theta_{(S \circ T)(1)_R} \otimes \theta_{(S \circ T)(1)_L})$, a toroidal compactification of the bilinear algebraic semigroups $G^{(n)}(F_v^+ \times F_v^+)$ and $G^{(n)}(F_{v_{\text{cov}}}^+ \times F_{v_{\text{cov}}}^+)$ on which they are defined must be performed in such a way that the products, right by left, of their corresponding conjugacy class representatives be products, right by left, of n -dimensional real semitori.

The bisemisheaves on the toroidal bilinear algebraic semigroups $G^{(n)}(F_v^{+,T} \times F_v^{+,T})$ and $G^{(n)}(F_{v_{\text{cov}}}^{+,T} \times F_{v_{\text{cov}}}^{+,T})$ will be written $(\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}})$ and $(\theta_{G_{T_R}^{(n)}}^{\text{cov}} \otimes \theta_{G_{T_L}^{(n)}}^{\text{cov}})$.

Remark that the toroidal compactifications of the bisemisheaves $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ and $(\theta_{(S \circ T)(1)_R} \otimes \theta_{(S \circ T)(1)_L})$ are such that their holomorphic representations are transformed into cuspidal representations according to:

$$\begin{array}{ccccc} \text{Irr hol}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}) & : & \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} & \longrightarrow & f_{\overline{v}}(z^*) \otimes f_v(z) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Irr ELLIP}(\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}) & : & \theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}} & \longrightarrow & \text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) \end{array}$$

where $\text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) = \text{ELLIP}_R(n, j_\delta, m_{j_\delta}) \otimes \text{ELLIP}_L(n, j_\delta, m_{j_\delta})$, being the global elliptic representation of $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$, given by the product, right by left, of n -dimensional

real global elliptic semimodules as introduced in [Pie1], corresponds to the searched cuspidal representation.

(Note that a similar procedure can be applied to the covering bisemisheaf $(\theta_{(S \circ T)(1)_R} \otimes S_{(S \circ T)(1)_L})$.)

We refer to proposition 4.2.10 which states the Langlands global correspondences as resulting from the singularization and the versal deformation of the bisemisheaf $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$.

In chapter 5, **the monodromy of (isolated) singularities** on the (bisemi)sheaf $(\theta_{G_R^{(n)}}^{\mathbb{C}} \otimes \theta_{G_L^{(n)}}^{\mathbb{C}})$ of differentiable bifunctions $\phi_{G_R^{(\mathbb{C})}}^{(n)}(z_{g_R}) \otimes \phi_{G_L^{(\mathbb{C})}}^{(n)}(z_{g_L})$ on the complex bilinear algebraic semigroup $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ is analysed and the Langlands global correspondences on the non singular fibres, generated by monodromy, are developed in the irreducible and reducible cases.

- The monodromy arises in an expanding phase which reflects the expansion of the subvarieties of a given variety with respect to a fixed measure and which is assumed to generate locally surjective morphisms of singularizations.
- The generated singularities can be non degenerate or degenerate in which case small deformations of these can split them up into simpler ones. So, assume that each section of the semisheaf $\theta_{G_L^{(n)}}^{\mathbb{R}} \subset \theta_{G_L^{(n)}}^{\mathbb{C}}$ (resp. $\theta_{G_R^{(n)}}^{\mathbb{R}} \subset \theta_{G_R^{(n)}}^{\mathbb{C}}$) is a Morse function affected by an isolated non degenerate singularity on a domain U_{j_L} (resp. U_{j_R}) included into the conjugacy class representative $g_L^{(n)}[j, m_j]$ (resp. $g_R^{(n)}[j, m_j]$) and described locally by

$$\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L}) = \sum_{i=1}^{2n} x_{i_{L_j}}^2 \quad (\text{resp.} \quad \phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R}) = \sum_{i=1}^{2n} x_{i_{R_j}}^2).$$

- The critical level set of $\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L})$ (resp. $\phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R})$) is the singular fibre $F_{\circ_{j_L}}^{(2n-1)}$ (resp. $F_{\circ_{j_R}}^{(2n-1)}$) given by

$$\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L}) = \sum_{i=1}^{2n} x_{i_{L_j}}^2 = 0 \quad (\text{resp.} \quad \phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R}) = \sum_{i=1}^{2n} x_{i_{R_j}}^2 = 0)$$

while the non singular fibres $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) are diffeomorphic to the space $TS_{L_j}^{2n-1}$ (resp. $TS_{R_j}^{2n-1}$) of the tangent bundle to a unit sphere $S_{L_j}^{2n-1}$ (resp. $S_{R_j}^{2n-1}$), which is diffeomorphic to the vanishing cycle $\Delta_{L_j}^{(2n-1)} \subset F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)} \subset F_{\lambda_{j_R}}^{(2n-1)}$).

As $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) is diffeomorphic to the unit sphere $S_{L_j}^{2n-1}$ (resp. $S_{R_j}^{2n-1}$), it must correspond to a function on the corresponding conjugacy class representative of the parabolic subgroup $P^{(2n-1)}(F_v^+)$ (resp. $P^{(2n-1)}(F_{\overline{v}}^+)$).

- So, the mapping:

$$\begin{aligned} h_{\gamma_{j_L}} : F_{\lambda_{j_L}}^{(2n-1)} &\longrightarrow F_{\lambda_{j_L}}^{(2n-1)} \\ (\text{resp. } h_{\gamma_{j_R}} : F_{\lambda_{j_R}}^{(2n-1)} &\longrightarrow F_{\lambda_{j_R}}^{(2n-1)}) \end{aligned}$$

of the non-singular fibre into itself is the monodromy of the closed loop $\gamma_{j_L} \subset \Delta_{L_j}^{(2n-1)}$ (resp. $\gamma_{j_R} \subset \Delta_{R_j}^{(2n-1)}$) realized by the conjugacy action of the j -th conjugacy class representative of the restricted linear algebraic semigroup $G^{(2n-1)}(F_{v_j}^{+(\text{res})})$ (resp. $G^{(2n-1)}(F_{\bar{v}_j}^{+(\text{res})})$).

- If a degenerate singularity decomposes by deformation into a set of elementary non degenerate singular points, the single monodromy becomes a monodromy group. In this context, if every section $\phi_{G_{j_L}^{(2n)}}^{(2n)}$ (resp. $\phi_{G_{j_R}^{(2n)}}^{(2n)}$), $1 \leq j \leq r \leq \infty$, of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$) is endowed with a set of k non degenerate singularities ω_{i_L} (resp. ω_{i_R}), $1 \leq i \leq k$, on U_{j_L} (resp. U_{j_R}), then the set of bisheaves $\{\mathcal{F}_{F_{i\lambda_{j_R}}^{(2n-1)}} \otimes \mathcal{F}_{F_{i\lambda_{j_L}}^{(2n-1)}}\}_{i=1}^k$ of non singular bifibres $F_{i\lambda_{j_R}}^{(2n-1)}(t) \otimes F_{i\lambda_{j_L}}^{(2n-1)}$ are generated by monodromy above every bisection of $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.
And, if there are b_i , $b_i \in \mathbb{N}$, non singular fibres in the sheaf $\mathcal{F}_{F_{i\lambda_{j_L}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{i\lambda_{j_R}}^{(2n-1)}}$), then we get a set of $k \times b_i$, $1 \leq i \leq k$, $1 \leq \beta_i \leq b_i$, monodromy bi(semi)sheaves above $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.
- Let $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{i,\beta_i}$, $1 \leq i \leq k$, $1 \leq \beta_i \leq b_i$, be the set of $k \times b_i$ monodromy bisemisheaves above the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.

Then a **global holomorphic correspondence** can be stated for the bismisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ as developed before and the set of global holomorphic correspondences:

$$\begin{aligned} \text{Irr Rep}_{W_{F_{R \times L}^+}^{\text{mon}}}^{(2n-1)}(W_{F_{R \text{mon}}^+}^{ab}(\beta_i) \times W_{F_{L \text{mon}}^+}^{ab}(\beta_i)) \\ \longrightarrow \text{Irr hol}^{(2n-1)}(\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)) \quad \forall i, \beta_i, \end{aligned}$$

can be similarly found for the monodromy bisemisheaves $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$.

After a toroidal compactification of these bisemisheaves, it is proved that:

- a) a **cuspidal representation**, given by the elliptic representation $\text{ELLIP}_{R \times L}(2n, j, m_j)$, can be associated with the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.

- b) **no cuspidal representation can be found for the monodromy bisemisheaves**, except if surgeries are performed.
- The **orthogonal reducibility of the bisemisheaf** $\theta_{\mathrm{GL}_{2n}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$ leads to the following decomposition:

$$\theta_{\mathrm{GL}_{2n=2_1+\dots+2_n}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega}) = \bigoplus_{\ell=1}^n \theta_{\mathrm{GL}_{2_\ell}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$$

where the irreducible bisemisheaves $\theta_{\mathrm{GL}_{2_\ell}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$ are able to generate monodromy groups.

If, on the domain, $U_{j_R}^{(2)} \times U_{j_L}^{(2)} \subset g_{R \times L}^{(2)}[j, m_j]$, each bifunction of $\theta_{\mathrm{GL}_{2_\ell}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$ is locally a Morse (bi)function in such a way that its critical set is the singular bifibre $F_{\circ_{j_R}}^{(1)} \times F_{\circ_{j_L}}^{(1)}$ given by

$$\phi_{G_{g_{j_R}}^{(\mathbb{C})}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(\mathbb{C})}}^{(2)}(U_{j_L}^{(2)}) = z_{j_1}^2 + z_{j_2}^2 = 0, \quad (z_1, z_2) \in \mathbb{C}^2,$$

then,

- 1) the corresponding non singular bifibres $F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$ are diffeomorphic to the product, right by left, $T_{\lambda_{j_R}}^2(t) \times T_{\lambda_{j_L}}^2(t)$ of two semitori.
- 2) the homology group $H_1(F_{\lambda_{j_L}}^{(1)}; \mathbb{Z}) \simeq \mathbb{Z}$ (resp. $H_1(F_{\lambda_{j_R}}^{(1)}; \mathbb{Z}) \simeq \mathbb{Z}$) of the semitorus $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) is generated by the upper (resp. lower) semicircle $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$) on $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) in such a way that $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$) shrinking onto the singularity, becomes the vanishing semicycle.

If each bisection of the bisemisheaf $\theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ is endowed with the same singular bifibre $F_{\circ_{j_R}}^{(1)} \times F_{\circ_{j_L}}^{(1)} = z_{j_1}^2 + z_{j_2}^2 = 0$, then a set of β bisemisheaves $\{\theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\mathrm{mon}}(b)\}_{b=1}^{\beta}$, isomorphic to (or “copies of”) the desingularized bisemisheaf $\theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$, can be generated by monodromy if β is the number of non singular bifibres above each bisection of $\theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.

And, a set of β **global holomorphic correspondences** can be associated with the β monodromy bisemisheaves according to:

$$\begin{array}{ccc} \mathrm{Irr} \, \mathrm{Rep}_{W_{F_{R \times L}}^{\mathrm{mon}}}^{(1)}(W_{F_{R \mathrm{mon}}}^{ab}(b) \times W_{F_{L \mathrm{mon}}}^{ab}(b)) & \longrightarrow & \mathrm{Irr} \, \mathrm{hol}^{(1)}(\theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\mathrm{mon}}(b)) \\ \parallel & & \\ \theta_{\mathrm{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\mathrm{mon}}(b) & \longrightarrow & f_{\overline{\omega}}(z_{m_b}^*) \times f_{\omega}(z_{m_b}), \quad 1 < b < \beta. \end{array}$$

Similarly, on the toroidal compactified monodromy bisemisheaves, the following **Langlands irreducible global correspondences can be stated:**

$$\begin{array}{ccc}
 \text{Irr } Rep_{W_{F_{R \times L}^{\text{mon}}}}^{(1)} (W_{F_{R \text{mon}}}^{ab}(b) \times W_{F_{L \text{mon}}}^{ab}(b)) & \longrightarrow & \text{Irr cusp}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})_{\text{mon}}}(b)) \\
 \parallel & & \\
 \theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})_{\text{mon}}}(b) & \longrightarrow & \text{EIS}_{R \times L}^{\text{mon}}(1, j, m_j)_b
 \end{array}$$

where $\text{EIS}_{R \times L}^{\text{mon}}(1, j, m_j)$, being the product, right by left, of the equivalents of the Eisenstein series, constitutes the cuspidal representation of the b -th monodromy bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})_{\text{mon}}}$.

1 (Bisemi)sheaf of differentiable (bi)functions on the bilinear algebraic semigroup $G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$

1.1 Completions at infinite places of a global number field

Let \tilde{F} denote a finite (algebraically closed) Galois extension of a global number field F^0 of characteristic zero. In the complex case, the splitting field $\tilde{F} = \tilde{F}_R \cup \tilde{F}_L$ is assumed to be composed of the left and right splitting semifields \tilde{F}_L and \tilde{F}_R in one-to-one correspondence in such a way that the left (resp. right) algebraic extension semifield \tilde{F}_L (resp. \tilde{F}_R) is the set of complex (resp. conjugate complex) simple roots of a polynomial ring over F^0 .

In the real case, the symmetric splitting field is noted $\tilde{F}^+ = \tilde{F}_R^+ \cup \tilde{F}_L^+$ where \tilde{F}_L^+ (resp. \tilde{F}_R^+) is the algebraic extension semifield composed of the set of positive (resp. symmetric negative) simple real roots.

The left and right equivalence classes of the local completions $F_L^{(+)}$ and $F_R^{(+)}$ respectively of $\tilde{F}_L^{(+)}$ and $\tilde{F}_R^{(+)}$ are the left and right complex (resp. real) infinite places of $F_L^{(+)}$ and $F_R^{(+)}$: they are noted $v = \{v_{1_\delta}, \dots, v_{j_\delta}, \dots, v_{t_\delta}\}$ and $\bar{v} = \{\bar{v}_{1_\delta}, \dots, \bar{v}_{j_\delta}, \dots, \bar{v}_{t_\delta}\}$ in the real case and $\omega = \{\omega_1, \dots, \omega_j, \dots, \omega_r\}$ and $\bar{\omega} = \{\bar{\omega}_1, \dots, \bar{\omega}_j, \dots, \bar{\omega}_r\}$ in the complex case and are equal in number.

The left (resp. right) complex pseudo-unramified completions $F_{\omega_j}^{nr}$ (resp. $F_{\bar{\omega}_j}^{nr}$), $1 \leq j \leq r$, of \tilde{F}_L (resp. \tilde{F}_R) are pseudo-unramified F^0 -semimodules characterized by their ranks, called global residue degrees,

$$[F_{\omega_j}^{nr} : F^0] = j \quad (\text{resp. } [F_{\bar{\omega}_j}^{nr} : F^0] = j),$$

and the left (resp. right) real pseudo-unramified completions $F_{v_{j_\delta}}^{+,nr}$ (resp. $F_{\bar{v}_{j_\delta}}^{+,nr}$), $1 \leq j_\delta \leq r$, of \tilde{F}_L (resp. \tilde{F}_R) are also characterized by their global residue degrees:

$$[F_{v_{j_\delta}}^{+,nr} : F^0] = j \quad (\text{resp. } [F_{\bar{v}_{j_\delta}}^{+,nr} : F^0] = j).$$

The left (resp. right) complex pseudo-ramified completions F_{ω_j} (resp. $F_{\bar{\omega}_j}$) of \tilde{F}_L (resp. \tilde{F}_R) and the left (resp. right) real pseudo-ramified completions $F_{v_{j_\delta}}^+$ (resp. $F_{\bar{v}_{j_\delta}}^+$) are F^0 -semimodules generated from irreducible central completions $F_{\omega_j^1}$ (resp. $F_{\bar{\omega}_j^1}$) of rank $N \cdot m^{(j_\delta)}$ in the complex case and from irreducible central completions $F_{v_{j_\delta}^1}^+$ (resp. $F_{\bar{v}_{j_\delta}^1}^+$) of rank N in the real case, where $m^{(j_\delta)}$ is the multiplicity of the j_δ -th real completion $F_{v_{j_\delta}^1}^+$ covering its complex equivalent F_{ω_j} .

So, if the irreducible central completions are given by their ranks:

- $[F_{\omega_j^1} : F^0] = N \cdot m^{(j_\delta)} \quad (\text{resp. } [F_{\bar{\omega}_j^1} : F^0] = N \cdot m^{(j_\delta)}),$
- $[F_{v_{j_\delta}^1}^+ : F^0] = N \quad (\text{resp. } [F_{\bar{v}_{j_\delta}^1}^+ : F^0] = N),$

the pseudo-ramified completions can be expressed from their corresponding pseudo-unramified equivalents as follows:

- $[F_{\omega_j} : F^0] = [F_{\omega_j}^{nr} : F^0] \times [F_{\omega_j^1} : F^0] = *_c + j \cdot N \cdot m^{(j_\delta)}$
 (resp. $[F_{\bar{\omega}_j} : F^0] = [F_{\bar{\omega}_j}^{nr} : F^0] \times [F_{\bar{\omega}_j^1} : F^0] = *_c + j \cdot N \cdot m^{(j_\delta)}).$
- $[F_{v_{j_\delta}} : F^0] = [F_{v_{j_\delta}}^{nr} : F^0] \times [F_{v_{j_\delta}^1} : F^0] = * + j \cdot N$
 (resp. $[F_{\bar{v}_{j_\delta}} : F^0] = [F_{\bar{v}_{j_\delta}}^{nr} : F^0] \times [F_{\bar{v}_{j_\delta}^1} : F^0] = * + j \cdot N),$

where $*_c$ denotes an integer inferior to $N \cdot m^{(j_\delta)}$ and $*$ an integer inferior to N .

Then, the complex pseudo-ramified completions F_{ω_j} (resp. $F_{\bar{\omega}_j}$), $1 \leq j \leq r \leq \infty$, can be approximatively cut into a set of j irreducible equivalent completions $F_{\omega_j^{j'}}$, $1 \leq j' \leq j$ (resp. $F_{\bar{\omega}_j^{j'}}$), of rank $N \cdot m^{(j_\delta)}$ while the real pseudo-ramified completions $F_{v_{j_\delta}}^+$ (resp. $F_{\bar{v}_{j_\delta}}^+$), $1 \leq j_\delta \leq r \leq \infty$, can be approximatively cut into a set of j irreducible equivalent completions $F_{v_{j_\delta}^{j'}}^+$, $1 \leq j' \leq j_\delta$ (resp. $F_{\bar{v}_{j_\delta}^{j'}}^+$), of rank N .

On the other hand, as a place is an equivalence class of completions, we have to take into account a set of complex completions $\{F_{\omega_j, m_j}\}_{m_j}$, $m_j \in \mathbb{N}$, $m_j \geq 1$, equivalent to the basic completion F_{ω_j} at the j -th complex place ω_j and characterized by the same rank as F_{ω_j} . These complex equivalent completions F_{ω_j, m_j} are generated from the basic completion F_{ω_j} in a nilpotent way [Pie1].

Similarly, at a real place v_{j_δ} , a set of real completions $\{F_{v_{j_\delta}, m_{j_\delta}}^+\}_{m_{j_\delta}}$ equivalent to the basic real completion $F_{v_{j_\delta}}^+$ and characterized by the same rank has to be considered.

As it was indicated before, each complex completion F_{ω_j} is covered by the set $\{F_{v_{j_\delta}, m_{j_\delta}}^+\}$ of $m^{(j_\delta)} = \sup(m_{j_\delta}) + 1$ real equivalent completions $F_{v_{j_\delta}, m_{j_\delta}}^+$.

Let

$$F_\omega = \{F_{\omega_1}, \dots, F_{\omega_j, m_j}, \dots, F_{\omega_r, m_r}\} \quad (\text{resp. } F_{\bar{\omega}} = \{F_{\bar{\omega}_1}, \dots, F_{\bar{\omega}_j, m_j}, \dots, F_{\bar{\omega}_r, m_r}\})$$

denote the set of complex pseudo-ramified completions at the set of complex places ω (resp. $\bar{\omega}$) and let

$$F_v^+ = \{F_{v_1}^+, \dots, F_{v_{j_\delta}, m_{j_\delta}}^+, \dots, F_{v_{r_\delta}, m_{r_\delta}}^+\} \quad (\text{resp. } F_{\bar{v}}^+ = \{F_{\bar{v}_1}^+, \dots, F_{\bar{v}_{j_\delta}, m_{j_\delta}}^+, \dots, F_{\bar{v}_{r_\delta}, m_{r_\delta}}^+\})$$

be the corresponding set of real pseudo-ramified completions at the set of real places v (resp. \bar{v}).

Then, the direct sum of the complex pseudo-ramified completions is given by:

$$F_{\omega_{\oplus}} = \bigoplus_j \bigoplus_{m_j} F_{\omega_j, m_j} \quad (\text{resp.} \quad F_{\bar{\omega}_{\oplus}} = \bigoplus_j \bigoplus_{m_j} F_{\bar{\omega}_j, m_j})$$

while the direct sum of the real pseudo-ramified completions is given by:

$$F_{v_{\oplus}} = \bigoplus_{j_{\delta}} \bigoplus_{m_{j_{\delta}}} F_{v_{j_{\delta}}, m_{j_{\delta}}} \quad (\text{resp.} \quad F_{\bar{v}_{\oplus}} = \bigoplus_{j_{\delta}} \bigoplus_{m_{j_{\delta}}} F_{\bar{v}_{j_{\delta}}, m_{j_{\delta}}}).$$

And, a left (resp. right) pseudo-ramified adele semiring $\mathbb{A}_{F_{\omega}}$ (resp. $\mathbb{A}_{F_{\bar{\omega}}}$) can be introduced on the product of the basic primary completions $F_{\omega_{j_p}}$ (resp. $F_{\bar{\omega}_{j_p}}$) and of their equivalent completions $F_{\omega_{j_p}, m_{j_p}}$ (resp. $F_{\bar{\omega}_{j_p}, m_{j_p}}$) over all primary complex places according to:

$$\mathbb{A}_{F_{\omega}} = \prod_{j_p} F_{\omega_{j_p}} \prod_{m_{j_p}} F_{\omega_{j_p}, m_{j_p}}, \quad 1 \leq j_p \leq r \leq \infty, \quad m_{j_p} \geq 1,$$

$$(\text{resp.} \quad \mathbb{A}_{F_{\bar{\omega}}} = \prod_{j_p} F_{\bar{\omega}_{j_p}} \prod_{m_{j_p}} F_{\bar{\omega}_{j_p}, m_{j_p}}).$$

Similarly, a left (resp. right) adele semiring $\mathbb{A}_{F_v^+}$ (resp. $\mathbb{A}_{F_{\bar{v}}^+}$) can be introduced over all primary real places according to:

$$\mathbb{A}_{F_v^+} = \prod_{j_{\delta p}} F_{v_{j_{\delta p}}}^+ \prod_{m_{j_{\delta p}}} F_{v_{j_{\delta p}}, m_{j_{\delta p}}}^+, \quad 1 \leq j_{\delta p} \leq r \leq \infty$$

$$(\text{resp.} \quad \mathbb{A}_{F_{\bar{v}}^+} = \prod_{j_{\delta p}} F_{\bar{v}_{j_{\delta p}}}^+ \prod_{m_{j_{\delta p}}} F_{\bar{v}_{j_{\delta p}}, m_{j_{\delta p}}}^+).$$

1.2 The reductive bilinear algebraic semigroup $G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$

The set F_{ω} (resp. $F_{\bar{\omega}}$) of complex pseudo-ramified completions generates a tower of r packets of completions following the complex places ω_j (resp. $\bar{\omega}_j$), $1 \leq j \leq r$: the left (resp. right) tower of r packets of complex pseudo-ramified completions, restricted to the upper (resp. lower) half space, is a one-dimensional complex linear affine semigroup noted $\mathbb{S}_{\omega_L}^1$ (resp. $\mathbb{S}_{\omega_R}^1$). For reasons developed in [Pie1], we are interested in the product, right by left, $\mathbb{S}_{\omega_R}^1 \times \mathbb{S}_{\omega_L}^1$ of $\mathbb{S}_{\omega_R}^1$ by its symmetric $\mathbb{S}_{\omega_L}^1$ where $\mathbb{S}_{\omega_R}^1 \times \mathbb{S}_{\omega_L}^1$ is a bilinear affine complex semigroup.

Similarly, a left (resp. right) tower of r packets of real pseudo-ramified completions is a one-dimensional real affine semigroup $\mathbb{S}_{v_L}^1$ (resp. $\mathbb{S}_{v_R}^1$) and the product, right by left, $\mathbb{S}_{v_R}^1 \times \mathbb{S}_{v_L}^1$ of $\mathbb{S}_{v_L}^1$ by $\mathbb{S}_{v_R}^1$ is a bilinear affine real semigroup.

The n -dimensional analog of $\mathbb{S}_{\omega_R}^1 \times \mathbb{S}_{\omega_L}^1$ is a n^2 -dimensional bilinear affine semigroup which is a reductive bilinear algebraic semigroup $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ isomorphic to the bilinear algebraic semigroup of matrices $\mathrm{GL}_n(F_{\overline{\omega}} \times F_{\omega})$ with entries in $F_{\overline{\omega}} \times F_{\omega}$. Indeed, $\mathrm{GL}_n(F_{\overline{\omega}} \times F_{\omega}) \equiv T_n^t(F_{\overline{\omega}}) \times T_n(F_{\omega})$ is a condensed notation for the product of the group $T_n^t(F_{\overline{\omega}})$ of lower triangular matrices with entries in $F_{\overline{\omega}}$ by the group $T_n(F_{\omega})$ of upper triangular matrices with entries in F_{ω} .

The bilinear algebraic semigroup $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ covers the corresponding linear algebraic group $G^{(n)}(F_{\overline{\omega}} - F_{\omega})$, where $F_{\overline{\omega}} - F_{\omega} = F_{\overline{\omega}} \cup F_{\omega}$, as it was justified in [Pie1], since the n^2 -dimensional representation space $M_{F_{\overline{\omega}}} \otimes M_{F_{\omega}}$ of $\mathrm{GL}_n(F_{\overline{\omega}} \times F_{\omega})$ then coincides with the n^2 -dimensional representation space V of $\mathrm{GL}_n(F_{\overline{\omega}} - F_{\omega})$ under some conditions given in [Pie1].

As the bilinear algebraic semigroup $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ is built over $F_{\overline{\omega}} \times F_{\omega}$, it is composed of r conjugacy classes, $1 \leq j \leq r$, having multiplicities $m^{(r)} = \sup(m_r) + 1$, where $m^{(r)}$ denotes the number of equivalent representatives in the r -th conjugacy class. Remark that the r conjugacy classes of $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ correspond to the r (bi)places of $F_{\overline{\omega}} \times F_{\omega}$.

1.3 Proposition

Let $\mathrm{SMOD}_{F_{\omega}}$ (resp. $\mathrm{SMOD}_{F_{\overline{\omega}}}$) denote the category of $T_n(F_{\omega})$ -semimodules $M_{F_{\omega}}$ (resp. $T_n^t(F_{\overline{\omega}})$ -semimodules $M_{F_{\overline{\omega}}}$) $\subset \mathrm{GL}_n(F_{\overline{\omega}} \times F_{\omega})$ -bisemimodules $M_{F_{\overline{\omega}}} \otimes M_{F_{\omega}}$.

Then, the $T_n(F_{\omega})$ -semimodule $M_{F_{\omega}}$ (resp. $T_n^t(F_{\overline{\omega}})$ -semimodule $M_{F_{\overline{\omega}}}$) is a division F_{ω} -semialgebra (resp. a division $F_{\overline{\omega}}$ -cosemialgebra).

Proof. Indeed, according to the appendix of [Pie1], the F_{ω} -semialgebra $M_{F_{\omega}}$ (resp. $F_{\overline{\omega}}$ -cosemialgebra $M_{F_{\overline{\omega}}}$) over the semiring F_{ω} (resp. $F_{\overline{\omega}}$) is a semiring $M_{F_{\omega}}$ (resp. $M_{F_{\overline{\omega}}}$) such that:

- a) $(M_{F_{\omega}}, +)$ (resp. $(M_{F_{\overline{\omega}}}, +)$) is a unitary left F_{ω} -semimodule (resp. right $F_{\overline{\omega}}$ -semimodule).
- b) $a_L (m_L n_L) = (a_L m_L) n_L = m_L (a_L n_L)$, $\forall a_L \in F_{\omega}$, $m_L, n_L \in M_{F_{\omega}}$
 (resp. $(m_R n_R) a_R = m_R (n_R a_R) = m_R (a_R n_R)$, $\forall a_R \in F_{\overline{\omega}}$, $m_R, n_R \in M_{F_{\overline{\omega}}}$).

As $M_{F_{\omega}}$ (resp. $M_{F_{\overline{\omega}}}$) is a left (resp. right) division semiring, $M_{F_{\omega}}$ (resp. $M_{F_{\overline{\omega}}}$) is a division F_{ω} -semialgebra (resp. $F_{\overline{\omega}}$ -cosemialgebra). ■

1.4 Definition : $F_{\overline{\omega}} \times F_{\omega}$ -bisemialgebra

- 1) The F_{ω} -semialgebra $M_{F_{\omega}}$ is a monoid $(M_{F_{\omega}}, \mu, \eta)$ in $\text{SMOD}_{F_{\omega}}$, in the sense that:
 - $M_{F_{\omega}}$ is assumed to be a unitary F_{ω} -semimodule, i.e. a left vector semispace over F_{ω} viewed as the center of $M_{F_{\omega}}$;
 - $\mu : M_{F_{\omega}} \otimes M_{F_{\omega}} \rightarrow M_{F_{\omega}}$ is a linear homomorphism;
 - $\eta : F_{\omega} \rightarrow M_{F_{\omega}}$ is an injective homomorphism.
- 2) The $F_{\overline{\omega}}$ -cosemialgebra $M_{F_{\overline{\omega}}}$ is dually a comonoid $(M_{F_{\overline{\omega}}}, \Delta, \varepsilon)$ in $\text{SMOD}_{F_{\overline{\omega}}}$ in such a way that:
 - $M_{F_{\overline{\omega}}}$ is assumed to be a unitary $M_{F_{\overline{\omega}}}$ -semimodule, i.e. a right vector semispace over $F_{\overline{\omega}}$ so that $M_{F_{\overline{\omega}}}$ is the dual of $M_{F_{\omega}}$;
 - $\Delta : M_{F_{\overline{\omega}}} \rightarrow M_{F_{\overline{\omega}}} \times M_{F_{\overline{\omega}}}$ is a linear homomorphism called comultiplication;
 - $\varepsilon : M_{F_{\overline{\omega}}} \rightarrow F_{\overline{\omega}}$ is a linear form.
- 3) $((M_{F_{\overline{\omega}}} \otimes M_{F_{\omega}}), \mu, \eta, \Delta, \varepsilon)$ is a division $(F_{\overline{\omega}} \times F_{\omega})$ -bisemialgebra if $(M_{F_{\omega}}, \mu, \eta)$ is a division F_{ω} -semialgebra and if $(M_{F_{\overline{\omega}}}, \Delta, \varepsilon)$ is a division $F_{\overline{\omega}}$ -cosemialgebra.

1.5 The bilinear parabolic subsemigroups

As the left (resp. right) complex pseudo-ramified completions F_{ω_j} (resp. $F_{\overline{\omega}_j}$) and the left (resp. right) real pseudo-ramified completions $F_{v_{j\delta}}^+$ (resp. $F_{\overline{v}_{j\delta}}^+$) are assumed to be generated respectively from irreducible central complex completions $F_{\omega_j^1}$ (resp. $F_{\overline{\omega}_j^1}$) of rank $N \cdot m^{(j\delta)}$ and from irreducible central real completions $F_{v_{j\delta}^1}^+$ (resp. $F_{\overline{v}_{j\delta}^1}^+$) of rank N , a set of left (resp. right) irreducible complex pseudo-ramified completions

$$F_{\omega^1} = \{F_{\omega_1^1}, \dots, F_{\omega_{j,m_j}^1}, \dots, F_{\omega_{r,m_r}^1}\}$$

$$(\text{resp. } F_{\overline{\omega}^1} = \{F_{\overline{\omega}_1^1}, \dots, F_{\overline{\omega}_{j,m_j}^1}, \dots, F_{\overline{\omega}_{r,m_r}^1}\})$$

can be introduced, as well as a set of left (resp. right) irreducible real pseudo-ramified completions:

$$F_{v^1}^+ = \{F_{v_{1\delta}^1}^+, \dots, F_{v_{j\delta,m_{j\delta}}^1}^+, \dots, F_{v_{r\delta,m_{r\delta}}^1}^+\}$$

$$(\text{resp. } F_{\overline{v}^1}^+ = \{F_{\overline{v}_{1\delta}^1}^+, \dots, F_{\overline{v}_{j\delta,m_{j\delta}}^1}^+, \dots, F_{\overline{v}_{r\delta,m_{r\delta}}^1}^+\}).$$

The n -dimensional smallest normal bilinear affine subsemigroup $G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$ (resp. $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$) is the parabolic bilinear algebraic subsemigroup $P^{(n)}(F_{\bar{\omega}}^1 \times F_{\omega}^1)$ (resp. $P^{(n)}(F_{\bar{v}}^+ \times F_v^+)$) isomorphic to the bilinear algebraic semigroup of matrices

$$\begin{aligned} \text{GL}_n(F_{\bar{\omega}^1} \times F_{\omega^1}) &= T_n^t(F_{\bar{\omega}^1}) \times T_n(F_{\omega^1}) \\ (\text{resp. } \text{GL}_n(F_{\bar{v}^1}^+ \times F_{v^1}^+) &= T_n^t(F_{\bar{v}^1}^+) \times T_n(F_{v^1}^+)) \end{aligned}$$

with entries in products of irreducible completions.

The parabolic bilinear subsemigroup $P^{(n)}(F_{\bar{\omega}}^1 \times F_{\omega}^1)$ (resp. $P^{(n)}(F_{\bar{v}}^+ \times F_v^+)$) can be considered as the unitary irreducible representation space of $\text{GL}_n(F_{\bar{\omega}} \times F_{\omega})$ (resp. $\text{GL}_n(F_{\bar{v}}^+ \times F_v^+)$), [Pie1].

1.6 The bialgebra of bifunctions on $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$

The smooth differentiable real-valued functions on the real bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ are the tensor products (called bifunctions)

$$\phi_{G_R}^{(n)}(x_{g_R}) \otimes \phi_{G_L}^{(n)}(x_{g_L})$$

of smooth differentiable functions $\phi_{G_L}^{(n)}(x_{g_L}) \in \widehat{G}_L^{(n)}(F_v^+)$, $x_{g_L} \in G_L^{(n)}(F_v^+)$, of the algebra $\widehat{G}_L^{(n)}(F_v^+)$ of these functions on the linear algebraic semigroup $G_L^{(n)}(F_v^+)$, localized in the upper half space, by the symmetric differentiable cofunctions $\phi_{G_R}^{(n)}(x_{g_R}) \in \widehat{G}_R^{(n)}(F_{\bar{v}}^+)$, $x_{g_R} \in G_R^{(n)}(F_{\bar{v}}^+)$, of the coalgebra $\widehat{G}_R^{(n)}(F_{\bar{v}}^+)$ of the cofunctions (or linear functionals) on the linear algebraic semigroup $G_R^{(n)}(F_{\bar{v}}^+)$, localized in the lower half space.

As $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is partitioned into conjugacy classes on the biplaces $\bar{v}_{j_{\delta}} \times v_{j_{\delta}}$, $1 \leq j_{\delta} \leq r$, of $F_R^+ \times F_L^+$, we have to consider bifunctions $\phi_{G_{j_R}}^{(n)}(x_{j_{\delta}}) \otimes \phi_{G_{j_L}}^{(n)}(x_{j_{\delta}})$ on the conjugacy class representatives $g_{R \times L}^{(n)}[j_{\delta}, m_{j_{\delta}}]$, also noted $G^{(n)}(F_{\bar{v}_{j_{\delta}, m_{j_{\delta}}}}^+ \times F_{v_{j_{\delta}, m_{j_{\delta}}}}^+)$, of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$.

As the conjugacy classes $g_{R \times L}^{(n)}[j_{\delta}]$ of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ form an increasing sequence

$$g_{R \times L}^{(n)}[1] \subset \cdots \subset g_{R \times L}^{(n)}[j_{\delta}] \subset \cdots \subset g_{R \times L}^{(n)}[r],$$

the bifunctions on the conjugacy class representatives $g_{R \times L}^{(n)}[j_{\delta}, m_{j_{\delta}}]$ also form an increasing sequence:

$$\phi_{G_{1_R}}^{(n)}(x_{g_{1_R}}) \otimes \phi_{G_{1_L}}^{(n)}(x_{g_{1_L}}) \subset \cdots \subset \phi_{G_{j_R}}^{(n)}(x_{g_{j_R}}) \otimes \phi_{G_{j_L}}^{(n)}(x_{g_{j_L}}) \subset \cdots.$$

The bialgebra of all differentiable real-valued measurable bifunctions $\phi_{G_R}^{(n)}(x_{g_R}) \otimes \phi_{G_L}^{(n)}(x_{g_L})$ on $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ satisfying:

$$\int_{G^{(n)}(F_{\bar{v}}^+ \times F_v^+)} \left| \phi_{G_R}^{(n)}(x_{g_R}) \otimes \phi_{G_L}^{(n)}(x_{g_L}) \right| dx_{g_R} dx_{g_L} < \infty$$

is noted $L_{R \times L}^{1-1}(G^{(n)}(F_v^+ \times F_v^+))$.

1.7 (Bisemi)sheaf of differentiable (bi)functions on $G^{(n)}(F_v^+ \times F_v^+)$

The set of real-valued differentiable functions $\phi_{G_L}^{(n)}(x_{g_L})$ (resp. cofunctions $\phi_{G_R}^{(n)}(x_{g_R})$) on the left (resp. right) linear algebraic semigroup $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) is a left (resp. right) semisheaf of rings $\theta_{G_L}^{(n)}$ (resp. $\theta_{G_R}^{(n)}$) because:

- a) for all basic conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta} = 0]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta} = 0]$) of the topological semispace $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$), we have a set $\theta_{G_L}^{(n)}(g_L^{(n)})[j_\delta, m_{j_\delta} = 0]$ (resp. $\theta_{G_R}^{(n)}(g_R^{(n)})[j_\delta, m_{j_\delta} = 0]$);
- b) for all pairs of basic conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta} = 0] \subset g_L^{(n)}[j_\delta + 1, m_{j_\delta+1} = 0]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta} = 0] \subset g_R^{(n)}[j_\delta + 1, m_{j_\delta+1} = 0]$) a restriction map

$$\begin{aligned} \text{res}_{g_L^{(n)}[j_\delta+1], g_L^{(n)}[j_\delta]} : \theta_{G_L}^{(n)}(g_L^{(n)}[j_\delta + 1]) &\longrightarrow \theta_{G_L}^{(n)}(g_L^{(n)}[j_\delta]) \\ (\text{resp. } \text{res}_{g_R^{(n)}[j_\delta+1], g_R^{(n)}[j_\delta]} : \theta_{G_R}^{(n)}(g_R^{(n)}[j_\delta + 1]) &\longrightarrow \theta_{G_R}^{(n)}(g_R^{(n)}[j_\delta])). \end{aligned}$$

a) and b) generate a presemisheaf of rings $\theta_{G_L}^{(n)}$ (resp. $\theta_{G_R}^{(n)}$) because it is a sheaf of abelian semigroups for every left (resp. right) point x_{g_L} (resp. x_{g_R}) of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) where $\theta_{G_L}^{(n)}(x_{g_L})$ (resp. $\theta_{G_R}^{(n)}(x_{g_R})$) has the structure of a semiring [Pie1], [Ser1].

The presemisheaf $\theta_{G_L}^{(n)}$ (resp. $\theta_{G_R}^{(n)}$) is a semisheaf of rings if for every collection $\{g_L^{(n)}[j_\delta]\}_{j=1}^r$ (resp. $\{g_R^{(n)}[j_\delta]\}_{j=1}^r$) of basic conjugacy class representatives in $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) with $g_L^{(n)} = \cup g_L^{(n)}[j_\delta]$ (resp. $g_R^{(n)} = \cup g_R^{(n)}[j_\delta]$), the map

$$\begin{aligned} \text{res}_{g_L^{(n)}} : \theta_{G_L}^{(n)}(g_L^{(n)}[j_\delta]) &\longrightarrow \prod_{j_\delta} \theta_{G_L}^{(n)}(g_L^{(n)}[j_\delta]) \\ (\text{resp. } \text{res}_{g_R^{(n)}} : \theta_{G_R}^{(n)}(g_R^{(n)}[j_\delta]) &\longrightarrow \prod_{j_\delta} \theta_{G_R}^{(n)}(g_R^{(n)}[j_\delta])) \end{aligned}$$

is injective [Mum].

The set of real valued differentiable bifunctions $\phi_{G_R}^{(n)}(x_{g_R}) \otimes \phi_{G_L}^{(n)}(x_{g_L})$ on $G^{(n)}(F_v^+ \times F_v^+)$ is a bisemisheaf of rings, noted $\theta_{G_{R \times L}}^{(n)} = \theta_{G_R}^{(n)} \otimes \theta_{G_L}^{(n)}$, whose bisections are the differentiable bifunctions $\phi_{G_{jR}}^{(n)}(x_{g_{j\delta}}) \otimes \phi_{G_{jL}}^{(n)}(x_{g_{j\delta}})$ on the conjugacy class representatives $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ of $G^{(n)}(F_v^+ \times F_v^+)$.

1.8 (Bi)ideal (bi)semisheaf of differentiable (bi)functions on the bilinear parabolic semigroup $P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+)$

Let $F_{v_{j_\delta}^1}^+$ (resp. $F_{\bar{v}_{j_\delta}^1}^+$) and $F_{v_{j_\delta, m_{j_\delta}}^1}^+$ (resp. $F_{\bar{v}_{j_\delta, m_{j_\delta}}^1}^+$) denote respectively the basic and the equivalent irreducible real completions of the j_δ -th real place v_{j_δ} (resp. \bar{v}_{j_δ}).

Let $P^{(n)}(F_{v_{j_\delta}^1}^+)$ (resp. $P^{(n)}(F_{\bar{v}_{j_\delta}^1}^+)$) be the n -dimensional left (resp. right) linear affine subsemigroup restricted to this j_δ -th basic irreducible completion: it is then the j_δ -th basic conjugacy class representative of $P^{(n)}(F_{v^1}^+)$ (resp. $P^{(n)}(F_{\bar{v}^1}^+)$).

On $P^{(n)}(F_{v_{j_\delta}^1}^+)$ (resp. $P^{(n)}(F_{\bar{v}_{j_\delta}^1}^+)$) we can introduce the ideal of complex-valued differentiable left (resp. right) functions $\phi_{P_{j_L}}^{(n)}(x_{p_{j_\delta}})$ (resp. $\phi_{P_{j_R}}^{(n)}(x_{p_{j_\delta}})$).

This ideal $\phi_{P_{j_L}}^{(n)}(x_{p_{j_\delta}})$ (resp. $\phi_{P_{j_R}}^{(n)}(x_{p_{j_\delta}})$) is an equivalence class of all real valued differentiable functions $\phi_{P_{j_L, m_{j_\delta}}}^{(n)}(x_{p_{j_\delta, m_{j_\delta}}})$ (resp. $\phi_{P_{j_R, m_{j_\delta}}}^{(n)}(x_{p_{j_\delta, m_{j_\delta}}})$) on the conjugacy class equivalent representatives $P^{(n)}(F_{v_{j_\delta, m_{j_\delta}}^1}^+)$ (resp. $P^{(n)}(F_{\bar{v}_{j_\delta, m_{j_\delta}}^1}^+)$).

So, the set of left (resp. right) ideals $\phi_{P_{j_L}}^{(n)}(x_{p_{j_\delta}})$ (resp. $\phi_{P_{j_R}}^{(n)}(x_{p_{j_\delta}})$) of real valued differentiable functions $\phi_{G_L}^{(n)}(x_{g_L})$ (resp. cofunctions $\phi_{G_R}^{(n)}(x_{g_R})$) is a left (resp. right) ideal semisheaf of differentiable functions (resp. cofunctions) on the parabolic semigroup $P^{(n)}(F_{v^1}^+)$ (resp. $P^{(n)}(F_{\bar{v}^1}^+)$).

And, the set of biideals $\{\phi_{P_{j_R}}^{(n)}(x_{p_{j_\delta}}) \otimes \phi_{P_{j_L}}^{(n)}(x_{p_{j_\delta}})\}$ of differentiable bifunctions is a biideal bisemisheaf, noted $\theta_{P_{R \times L}^{(n)}} = \theta_{P_R^{(n)}} \otimes \theta_{P_L^{(n)}}$, whose bisections are the biideals on the conjugacy class representatives $P^{(n)}(F_{\bar{v}_{j_\delta}^1}^+ \times F_{v_{j_\delta}^1}^+)$ of the bilinear parabolic semigroup $P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+)$.

1.9 (Bisemi)sheaf of complex valued differentiable (bi)functions on $G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$

Considering the inclusion

$$G^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+) \hookrightarrow G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$$

of the real bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+)$ into the corresponding complex bilinear algebraic semigroup $G^{(n)}(F_{\bar{\omega}} \times F_{\omega})$ as envisaged in [Pie1], we can introduce the semisheaf $\theta_{G_L}^{(\mathbb{C})}$ (resp. $\theta_{G_R}^{(\mathbb{C})}$) of complex valued differentiable functions $\phi_{G_L}^{(\mathbb{C})}(x_{g_L})$ (resp. cofunctions $\phi_{G_R}^{(\mathbb{C})}(x_{g_R})$) on the left (resp. right) linear algebraic semigroup $G^{(n)}(F_{\omega})$ (resp. $G^{(n)}(F_{\bar{\omega}})$). $\theta_{G_L}^{(\mathbb{C})}$ (resp. $\theta_{G_R}^{(\mathbb{C})}$) can be defined similarly as it was done in section 1.7 and, in the following, the real case will be essentially considered.

2 singularization and versal deformation

2.1 The singularization

2.1.1 General statement of the singularization process

The left (resp. right) semisheaf $\Theta_{G_L^{(n)}}$ (resp. $\Theta_{G_R^{(n)}}$) is a semisheaf of smooth differentiable functions on the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) of the left (resp. right) real linear algebraic semigroup $G^{(n)}(F_v^+) \simeq T_n(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+) \simeq T_n^t(F_{\bar{v}}^+)$).

Under some external perturbation(s), singularities can be generated on the left (resp. right) semisheaf $\Theta_{G_L^{(n)}}$ (resp. $\Theta_{G_R^{(n)}}$) in such a way that:

- a) these singularities are produced symmetrically on the functions and cofunctions respectively on the left and on the right conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}] \in G^{(n)}(F_v^+)$ and $g_R^{(n)}[j_\delta, m_{j_\delta}] \in G^{(n)}(F_{\bar{v}}^+)$: this results from the fact that $G^{(n)}(F_v^+)$ and $G^{(n)}(F_{\bar{v}}^+)$ are symmetrical algebraic semigroups localized in some small domains respectively in the upper and in the lower half space. So, an external perturbation affects in a similar way functions on the upper half space and cofunctions on the lower half space.
- b) on each function $\phi_{G_{j_L}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ on $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. cofunction $\phi_{G_{j_R}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ on $g_R^{(n)}[j_\delta, m_{j_\delta}]$), a same singularity (or a same set of singularities) is generated.

Indeed, according to a), the external perturbation is assumed to affect similarly and symmetrically every function $\phi_{G_{j_L}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ and cofunction $\phi_{G_{j_R}^{(n)}}^{(n)}(x_{g_{j_\delta}})$.

The process of generation of singularities will be called a singularization. It consists of a collapse of (a) normal crossings divisor(s) into a locus becoming singular: this is a contracting surjective morphism corresponding to the inverse of a resolution of singularities (see, for example, [Abh], [Ber], [DeJ], [Hir1, Hir2, Hir3], [Zar1, Zar2, Zar3]).

2.1.2 Definition: singularization of regular f -schemes

The left (resp. right) linear algebraic semigroup $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+)$) plus the left (resp. right) semisheaf $\Theta_{G_L^{(n)}}$ (resp. $\Theta_{G_R^{(n)}}$) on it is a left (resp. right) affine semischeme $(G^{(n)}(F_v^+), \Theta_{G_L^{(n)}})$ (resp. $(G^{(n)}(F_{\bar{v}}^+), \Theta_{G_R^{(n)}})$) [Mum].

Every differentiable left (resp. right) function $\phi_{G_{j_L}^{(n)}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}^{(n)}}^{(n)}(x_{g_{j_\delta}})$) of $\Theta_{G_L^{(n)}}$ (resp. $\Theta_{G_R^{(n)}}$) being similarly affected by some external perturbation, will be considered

as a “prototype” left (resp. right) scheme function, written in condensed form ϕ_L (resp. ϕ_R) and called a left (resp. right) f -scheme.

The singularization of the left (resp. right) regular f -scheme ϕ_L (resp. ϕ_R) is a contracting surjective morphism

$$\begin{aligned} \rho_L : \quad \phi_L &\rightarrow \phi_L^* \quad (* \text{ is for “star” symbolizing the singularities}) \\ (\text{resp. } \rho_R : \quad \phi_R &\rightarrow \phi_R^*) \end{aligned}$$

yielding a left (resp. right) singular f -scheme ϕ_L^* (resp. ϕ_R^*) below ϕ_L (resp. ϕ_R) and verifying:

- 1) ϕ_L (resp. ϕ_R) and ϕ_L^* (resp. ϕ_R^*) have the same dimension n .
- 2) the singular f -scheme ϕ_L^* (resp. ϕ_R^*) is characterized by a singular locus Σ_L (resp. Σ_R) associated with the centre Z_L (resp. Z_R) of the singularization of ϕ_L^* (resp. ϕ_R^*).

2.1.3 Proposition

The inverse morphism

$$\rho_L^{-1} : \quad \phi_L^* \rightarrow \phi_L, \quad (\text{resp. } \rho_R^{-1} : \quad \phi_R^* \rightarrow \phi_R)$$

of the singularization of the left (resp. right) regular f -scheme ϕ_L (resp. ϕ_R) corresponds to the desingularization of ϕ_L^ (resp. ϕ_R^*) if there exists an inclusion:*

$$h_L : \quad \phi_L \hookrightarrow \bar{\phi}_L \quad (\text{resp. } h_R : \quad \phi_R \hookrightarrow \bar{\phi}_R)$$

such that:

- 1) $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) is a regular projective left (resp. right) f -scheme.
- 2) $\rho_L^{-1}(Z_L) \cup (\bar{\phi}_L \setminus \phi_L)$ (resp. $\rho_R^{-1}(Z_R) \cup (\bar{\phi}_R \setminus \phi_R)$) is a closed f -subscheme of $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) identified with a normal crossings divisor D_L (resp. D_R).

Proof. The normal crossings divisor D_L (resp. D_R) is a regular closed f -subscheme of $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) and is the image under $\rho_L^{-1} \circ h_L$ (resp. $\rho_R^{-1} \circ h_R$) (of the centre Z_L (resp. Z_R)) of the singular locus Σ_L (resp. Σ_R) above which ρ_L (resp. ρ_R) is not an isomorphism. The centre Z_L (resp. Z_R) is generally in the singular locus Σ_L (resp. Σ_R) but, up to now, no ad hoc definition of the centre working in any dimension has been discovered [Hau]. ■

2.1.4 Definition: Normal crossings divisor

- A divisor has normal crossings if it can be defined locally by a monomial ideal.
- A normal crossings divisor D_L (resp. D_R) will be assumed to be a closed f -subscheme function on one or on a set of real irreducible completions $F_{v_{j\delta}}^+$ (resp. $F_{\bar{v}_{j\delta}}^+$) of rank N (see section 1.1).

2.1.5 Proposition (singularization)

Let D_L (resp. D_R) be a normal crossings divisor included into the left (resp. right) regular f -scheme $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) in such a way that:

$$\bar{\phi}_L = \phi_L \cup D_L \quad (\text{resp.} \quad \bar{\phi}_R = \phi_R \cup D_R).$$

The singularization of $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) into a singular locus Σ_L (resp. Σ_R) is given by the contracting surjective morphism:

$$\bar{\rho}_L : \bar{\phi}_L \rightarrow \phi_L^* \quad (\text{resp.} \quad \bar{\rho}_R : \bar{\phi}_R \rightarrow \phi_R^*)$$

such that:

- a) $\Sigma_L \subset \phi_L^*$ (resp. $\Sigma_R \subset \phi_R^*$) be the union of the (homotopic) image of $D_L \subset \bar{\phi}_L$ (resp. $D_R \subset \bar{\phi}_R$) and of a closed singular sublocus $\Sigma_L^S \subset \Sigma_L$ (resp. $\Sigma_R^S \subset \Sigma_R$) of ϕ_L^* (resp. ϕ_R^*):

$$\begin{aligned} \Sigma_L &= \bar{\rho}_L(D_L) \cup \Sigma_L^S \\ (\text{resp.} \quad \Sigma_R &= \bar{\rho}_R(D_R) \cup \Sigma_R^S). \end{aligned}$$

- b) $\bar{\rho}_L$ (resp. $\bar{\rho}_R$) restricted to:

$$\begin{aligned} \bar{\rho}_L^{\text{is}} : \phi_L \setminus \bar{\rho}_L^{-1}(\Sigma_L^S) &\longrightarrow \phi_L^* \setminus \Sigma_L \\ (\text{resp.} \quad \bar{\rho}_R^{\text{is}} : \phi_R \setminus \bar{\rho}_R^{-1}(\Sigma_R^S) &\longrightarrow \phi_R^* \setminus \Sigma_R) \end{aligned}$$

be an isomorphism.

Proof.

- The centre Z_L (resp. Z_R) corresponds to the singular locus Σ_L (resp. Σ_R) in the case of curves because their singularities are isolated points.

For surfaces, the situation is more complicated because the singular locus consists of a finite number of isolated points and irreducible curves which may be singular [Hau].

But generally, $Z_L \subset \bar{\rho}_L(D_L)$ (resp. $Z_R \subset \bar{\rho}_R(D_R)$).

- On the other hand, the singular locus Σ_L (resp. Σ_R) may factorize into:

$$\Sigma_L = E_L \cdot I_L \quad (\text{resp. } \Sigma_R = E_R \cdot I_R)$$

where:

- E_L (resp. E_R) is a power of the exceptional component $\bar{\rho}_L(D_L)$ (resp. $\bar{\rho}_R(D_R)$), i.e. a monomial;
- I_L (resp. I_R) is an ideal which has at each point of Σ_L (resp. Σ_R) order less than or equal to the order of $\bar{\rho}_L(D_L)$ (resp. $\bar{\rho}_R(D_R)$) along Z_L (resp. Z_R).
- I_L (resp. I_R) is the weak transform of $\bar{\rho}_L(D_L)$ (resp. $\bar{\rho}_R(D_R)$).
- The singular sublocus $\Sigma_L^S \subset \phi_L^*$ (resp. $\Sigma_R^S \subset \phi_R^*$) results from singularizations anterior to that of $\bar{\rho}_L(D_L)$ (resp. $\bar{\rho}_R(D_R)$) and then becomes the singular locus of the following blowups of $\phi_L^* \setminus \bar{\phi}_L(D_L)$ (resp. $\phi_R^* \setminus \bar{\phi}_R(D_R)$): this corresponds to a more general case than envisaged in proposition 2.1.3.
- Let:

$$\begin{aligned} \bar{\rho}_L^S : D_L &\longrightarrow \Sigma_L \setminus \Sigma_L^S \\ (\text{resp. } \bar{\rho}_R^S : D_R &\longrightarrow \Sigma_R \setminus \Sigma_R^S) \end{aligned}$$

be the singularization map restricted to the singular locus $\Sigma_L \setminus \Sigma_L^S$ (resp. $\Sigma_R \setminus \Sigma_R^S$).

Then, $\Sigma_L \setminus \Sigma_L^S$ (resp. $\Sigma_R \setminus \Sigma_R^S$) is the contracting homotopic image of D_L (resp. D_R) in the sense that the number n_{D_L} (resp. n_{D_R}) of irreducible real completions on which is defined D_L (resp. D_R) is superior or equal to the number $n_{(\Sigma_L \setminus \Sigma_L^S)}$ (resp. $n_{(\Sigma_R \setminus \Sigma_R^S)}$) of irreducible real completions on which is defined $\Sigma_L \setminus \Sigma_L^S$ (resp. $\Sigma_R \setminus \Sigma_R^S$):

$$n_{D_L} \leq n_{(\Sigma_L \setminus \Sigma_L^S)} \quad (\text{resp. } n_{D_R} \leq n_{(\Sigma_R \setminus \Sigma_R^S)}). \quad \blacksquare$$

2.1.6 Proposition

The singularization $\bar{\rho}_L : \bar{\phi}_L \rightarrow \phi_L^*$ (resp. $\bar{\rho}_R : \bar{\phi}_R \rightarrow \phi_R^*$) of $\bar{\phi}_L$ (resp. $\bar{\phi}_R$) is assumed to be given by the following sequence of contracting surjective morphisms:

$$\begin{aligned} \bar{\phi}_L \equiv \phi_L^{*(0)} &\xrightarrow{\bar{\rho}_L^{(1)}} \phi_L^{*(1)} \xrightarrow{\bar{\rho}_L^{(2)}} \phi_L^{*(2)} \longrightarrow \dots \xrightarrow{\bar{\rho}_L^{(r-1)}} \phi_L^{*(r-1)} \xrightarrow{\bar{\rho}_L^{(r)}} \phi_L^{*(r)} \\ (\text{resp. } \bar{\phi}_R \equiv \phi_R^{*(0)} &\xrightarrow{\bar{\rho}_R^{(1)}} \phi_R^{*(1)} \xrightarrow{\bar{\rho}_R^{(2)}} \phi_R^{*(2)} \longrightarrow \dots \xrightarrow{\bar{\rho}_R^{(r-1)}} \phi_R^{*(r-1)} \xrightarrow{\bar{\rho}_R^{(r)}} \phi_R^{*(r)}) \end{aligned}$$

verifying

1) the singular locus

$$\Sigma_L \subset \phi_L^* \equiv \phi_L^{*(r)} \quad (\text{resp. } \Sigma_R \subset \phi_R^* \equiv \phi_R^{*(r)})$$

is given by

$$\begin{aligned} \Sigma_L &\equiv \Sigma_L^{(r)} = \Sigma_L^{(1)} \cup \bar{\rho}_L^{(2)}(D_L^{(1)}) \cup \dots \cup \bar{\rho}_L^{(\ell)}(D_L^{(\ell-1)}) \dots \cup \dots \cup \bar{\rho}_L^{(r)}(D_L^{(r-1)}) \\ (\text{resp. } \Sigma_R &\equiv \Sigma_R^{(r)} = \Sigma_R^{(1)} \cup \bar{\rho}_R^{(2)}(D_R^{(1)}) \cup \dots \cup \bar{\rho}_R^{(\ell)}(D_R^{(\ell-1)}) \dots \cup \dots \cup \bar{\rho}_R^{(r)}(D_R^{(r-1)})). \end{aligned}$$

2) $\bar{\rho}_L$ (resp. $\bar{\rho}_R$) restricted to:

$$\begin{aligned} \bar{\rho}_L^{(\text{is})} : \quad \bar{\phi}_L \setminus \bar{\rho}_L^{-1}(\Sigma_L^{(r)}) &\longrightarrow \phi_L^* \setminus \Sigma_L^{(r)} \\ (\text{resp. } \bar{\rho}_R^{(\text{is})} : \quad \bar{\phi}_R \setminus \bar{\rho}_R^{-1}(\Sigma_R^{(r)}) &\longrightarrow \phi_R^* \setminus \Sigma_R^{(r)}) \end{aligned}$$

is an isomorphism.

3) The orders of the singular subloci $\Sigma_L^{(\ell)}$ (resp. $\Sigma_R^{(\ell)}$) form an increasing sequence (from left to right) parallely with the increase of (ℓ) , $1 \leq \ell \leq r$.

Proof.

- Let $\bar{\rho}_L^{(\text{sing})}$ (resp. $\bar{\rho}_R^{(\text{sing})}$) denote the complement of $\bar{\rho}_L^{(\text{is})}$ (resp. $\bar{\rho}_R^{(\text{is})}$) in $\bar{\rho}_L$ (resp. $\bar{\rho}_R$) : it is defined by the increasing sequence of subloci:

$$\begin{aligned} \bar{\rho}_L^{(\text{sing})} : \quad D_L^{(0)} &\xrightarrow{\bar{\rho}_L^{(1)}} \Sigma_L^{(1)} \cup D_L^{(1)} \xrightarrow{\bar{\rho}_L^{(2)}} \Sigma_L^{(2)} \cup D_L^{(2)} \\ &\longrightarrow \dots \xrightarrow{\bar{\rho}_L^{(\ell)}} \Sigma_L^{(\ell)} \cup D_L^{(\ell)} \longrightarrow \dots \xrightarrow{\bar{\rho}_L^{(r)}} \Sigma_L^{(r)} \\ (\text{resp. } \bar{\rho}_R^{(\text{sing})} : \quad D_R^{(0)} &\xrightarrow{\bar{\rho}_R^{(1)}} \Sigma_R^{(1)} \cup D_R^{(1)} \xrightarrow{\bar{\rho}_R^{(2)}} \Sigma_R^{(2)} \cup D_R^{(2)} \\ &\longrightarrow \dots \xrightarrow{\bar{\rho}_R^{(\ell)}} \Sigma_R^{(\ell)} \cup D_R^{(\ell)} \longrightarrow \dots \xrightarrow{\bar{\rho}_R^{(r)}} \Sigma_R^{(r)}) \end{aligned}$$

where

- $\Sigma_L^{(\ell)} = \Sigma_L^{(\ell-1)} \cup \bar{\rho}_L^{(\ell)}(D_L^{(\ell-1)})$ (resp. $\Sigma_R^{(\ell)} = \Sigma_R^{(\ell-1)} \cup \bar{\rho}_R^{(\ell)}(D_R^{(\ell-1)})$);
- $\Sigma_L^{(\ell)}$ (resp. $\Sigma_R^{(\ell)}$) is the ℓ -th singular sublocus generated by the composition of contracting surjective morphisms:

$$\bar{\rho}_L^{(1)} \circ \dots \circ \bar{\rho}_L^{(\ell-1)} \circ \bar{\rho}_L^{(\ell)} \quad (\text{resp. } \bar{\rho}_R^{(1)} \circ \dots \circ \bar{\rho}_R^{(\ell-1)} \circ \bar{\rho}_R^{(\ell)})$$

restricted to their action on the normal crossings divisors $D_L^{(0)} \dots D_L^{(\ell-1)}$ (resp. $D_R^{(0)} \dots D_R^{(\ell-1)}$).

- Let $\Sigma_L^{(\ell-1)} = E_L^{(\ell-1)} \cdot I_L^{(\ell-1)}$ (resp. $\Sigma_R^{(\ell-1)} = E_R^{(\ell-1)} \cdot I_R^{(\ell-1)}$) be the factorisation of the singular sublocus $\Sigma_L^{(\ell-1)}$ (resp. $\Sigma_R^{(\ell-1)}$) as introduced in proposition 2.1.5.

And, let

$$\begin{aligned} \bar{\rho}_L^{(\text{sing})^{(\ell)}} : \Sigma_L^{(\ell-1)} \cup D_L^{(\ell-1)} &\longrightarrow \Sigma_L^{(\ell)} \\ \text{(resp. } \bar{\rho}_R^{(\text{sing})^{(\ell)}} : \Sigma_R^{(\ell-1)} \cup D_R^{(\ell-1)} &\longrightarrow \Sigma_R^{(\ell)} \end{aligned}$$

be the ℓ -th surjective morphism restricted to the singular sublocus $\Sigma_L^{(\ell)}$ (resp. $\Sigma_R^{(\ell)}$): $\bar{\rho}_L^{(\text{sing})^{(\ell)}}$ (resp. $\bar{\rho}_R^{(\text{sing})^{(\ell)}}$) is a fibre bundle with fibre $D_L^{(\ell-1)}$ (resp. $D_R^{(\ell-1)}$).

Then, the order of $\Sigma_L^{(\ell)}$ (resp. $\Sigma_R^{(\ell)}$) at each point p_L (resp. p_R), being the maximal power of the maximal ideal of p_L (resp. p_R), is superior to the order of $\Sigma_L^{(\ell-1)}$ (resp. $\Sigma_R^{(\ell-1)}$).

Indeed, the factorisations of $\Sigma_L^{(\ell-1)}$ (resp. $\Sigma_R^{(\ell-1)}$) and of $\Sigma_L^{(\ell)}$ (resp. $\Sigma_R^{(\ell)}$) are respectively given by:

$$\Sigma_L^{(\ell-1)} = E_L^{(\ell-1)} \cdot I_L^{(\ell-1)} \quad (\text{resp. } \Sigma_R^{(\ell-1)} = E_R^{(\ell-1)} \cdot I_R^{(\ell-1)})$$

and by

$$\Sigma_L^{(\ell)} = E_L^{(\ell)} \cdot I_L^{(\ell)} \quad (\text{resp. } \Sigma_R^{(\ell)} = E_R^{(\ell)} \cdot I_R^{(\ell)})$$

in such a way that the order of $E_L^{(\ell)}$ (resp. $E_R^{(\ell)}$) is superior or equal to the order of $E_L^{(\ell-1)}$ (resp. $E_R^{(\ell-1)}$), taking into account [Hau] that, at the beginning,

$$\Sigma_L^{(0)} \equiv D_L^{(0)} = E_L^{(0)} \cdot I_L^{(0)} \quad (\text{resp. } \Sigma_R^{(0)} \equiv D_R^{(0)} = E_R^{(0)} \cdot I_R^{(0)})$$

is such that $E_L^{(0)} = 1$ (resp. $E_R^{(0)} = 1$) and $D_L^{(0)} = I_L^{(0)}$ (resp. $D_R^{(0)} = I_R^{(0)}$). ■

2.1.7 Definition: Corank of the singular locus

Let $P(x_L, y_L, z_L)$ (resp. $P(x_R, y_R, z_R)$) be the polynomial characterizing the singular locus Σ_L (resp. Σ_R). The number of variables of $P(x_L, y_L, z_L)$ (resp. $P(x_R, y_R, z_R)$) is the corank of Σ_L (resp. Σ_R).

This corank is inferior or equal to 3 according to [A-V-G1].

2.1.8 Ideals of differentiable functions

From the beginning of chapter 2, differentiable left (resp. right) functions of $\phi_{G_{jL}}^{(n)}(x_{g_{j\delta}})$ (resp. $\phi_{G_{jR}}^{(n)}(x_{g_{j\delta}})$) on the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j\delta}]$) were taken into account (see section 2.1.1).

If we consider the set $\{\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}})\}_{m_{j_\delta}}$ (resp. $\{\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}})\}_{m_{j_\delta}}$) of differentiable functions on the j_δ -th conjugacy class of $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+)$) restricted to the singular loci $\Sigma_L[j_\delta, m_{j_\delta}]$ (resp. $\Sigma_R[j_\delta, m_{j_\delta}]$), we introduce an ideal of differentiable functions characterized by the polynomial $P_{j_R}(x_L, y_L, z_L)$ (resp. $P_{j_L}(x_R, y_R, z_R)$) (see section 1.8).

2.1.9 Simple germs of differentiable functions

Assume that the singular locus $\Sigma_L[j_\delta, m_{j_\delta}]$ (resp. $\Sigma_R[j_\delta, m_{j_\delta}]$) is given by a singular point of finite codimension. Then, the corresponding simple germs of differentiable functions are the following [A-V-G1]:

$$\begin{aligned} A_K : \quad f(x) &= x^{k+1}, & k \geq 1, \\ D_K : \quad f(x, y) &= x^2 y + y^{k-1}, & k \geq 4, \\ E_6 : \quad f(x, y) &= x^3 + y^4, \\ E_7 : \quad f(x, y) &= x^3 + x y^3, \\ E_8 : \quad f(x, y) &= x^3 + y^5. \end{aligned}$$

They are described by the classical Dynkin diagrams.

Applying proposition 2.1.6, we shall now see how it is possible to generate a singular point of corank 1 and multiplicity k by a singularization of type A_k .

2.1.10 Proposition

A singularization of type A_k , given by the germ $y_L = x_L^{k+1}$ (resp. $y_R = x_R^{k+1}$) of differentiable functions $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}})$) on the j_δ -th conjugacy class of $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+)$), is generated by the following sequence of contracting surjective morphisms:

$$\begin{aligned} \bar{\rho}_L^{(\text{sing})} : \quad D_L^{(0)} \xrightarrow{\bar{\rho}_L^{(1)}} x_L \cup D_L^{(1)} \longrightarrow \dots \xrightarrow{\bar{\rho}_L^{(k)}} x_L^k \cup D_L^{(k)} \xrightarrow{\bar{\rho}_L^{(k+1)}} x_L^{k+1} \\ (\text{resp. } \bar{\rho}_R^{(\text{sing})} : \quad D_R^{(0)} \xrightarrow{\bar{\rho}_R^{(1)}} x_R \cup D_R^{(1)} \longrightarrow \dots \xrightarrow{\bar{\rho}_R^{(k)}} x_R^k \cup D_R^{(k)} \xrightarrow{\bar{\rho}_R^{(k+1)}} x_R^{k+1}) \end{aligned}$$

restricted to the singular subloci $\Sigma_L^{(k)} = x_L^k$ (resp. $\Sigma_R^{(k)} = x_R^k$), $1 \leq k \leq k+1$, in such a way that:

1) the contracting surjective morphism:

$$\begin{aligned} \bar{\rho}_L^{(k)} : \quad x_L^{k-1} \cup D_L^{(k-1)} \longrightarrow x_L^k, & \quad 1 \leq k \leq k+1, \\ (\text{resp. } \bar{\rho}_R^{(k)} : \quad x_R^{k-1} \cup D_R^{(k-1)} \longrightarrow x_R^k), \end{aligned}$$

restricted to the singular sublocus $\Sigma_L^{(k)}$ (resp. $\Sigma_R^{(k)}$), is a fibre bundle whose fibre $D_L^{(k-1)}$ (resp. $D_R^{(k-1)}$), being a normal crossings divisor on a real irreducible completion $F_{v_{j_\delta}^+}^{+}$ (resp. $F_{v_{j_\delta}^+}^{+}$) according to definition 2.1.4, collapses into the germ x_L^k (resp. x_R^k). The fibre $D_L^{(k-1)}$ (resp. $D_R^{(k-1)}$) is thus a contracting fibre).

$$2) \quad \begin{aligned} \bar{\rho}_L : \quad & \bar{\phi}_{G_{j_L}}^{(n)}(x_{g_{j_\delta}}) \cup (D_L^{(0)}, \dots, D_L^{(k)}) \longrightarrow \phi_{G_{j_L}}^{*(n)}(x_{g_{j_\delta}}) \\ (\text{resp. } \bar{\rho}_R : \quad & \bar{\phi}_{G_{j_R}}^{(n)}(x_{g_{j_\delta}}) \cup (D_R^{(0)}, \dots, D_R^{(k)}) \longrightarrow \phi_{G_{j_R}}^{*(n)}(x_{g_{j_\delta}}) \end{aligned}$$

is the singularization morphism of the differentiable function $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}})$) generating a germ $y_L = x_L^{k+1}$ (resp. $y_R = x_R^{k+1}$) on it.

$$3) \quad \begin{aligned} \bar{\rho}_L^{(\text{is})} : \quad & \bar{\phi}_{G_{j_L}}^{(n)}(x_{g_{j_\delta}}) \setminus \bar{\rho}_L^{-1}(\Sigma_L^{(k+1)} = x_L^{k+1}) \rightarrow \phi_{G_{j_L}}^{*(n)}(x_{g_{j_\delta}}) \setminus (\Sigma_L^{(k+1)} = x_L^{k+1}) \\ (\text{resp. } \bar{\rho}_R^{(\text{is})} : \quad & \bar{\phi}_{G_{j_R}}^{(n)}(x_{g_{j_\delta}}) \setminus \bar{\rho}_R^{-1}(\Sigma_R^{(k+1)} = x_R^{k+1}) \rightarrow \phi_{G_{j_R}}^{*(n)}(x_{g_{j_\delta}}) \setminus (\Sigma_R^{(k+1)} = x_R^{k+1}) \end{aligned}$$

is an isomorphism.

2.1.11 Corollary

Every contractive surjective morphism $\bar{\rho}_L^{(k)}$ of the sequence $\bar{\rho}_L^{(\text{sing})}$ of proposition 2.1.10 provides a germ $y_L = x_L^{k+1}$ of type A_k at a singular point of corank 1 and multiplicity k in such a way that this sequence $\bar{\rho}_L^{(\text{sing})}$ of singularizations generates the following sequence of simple germs:

$$x_L^2 \subset x_L^3 \subset \dots \subset x_L^k \subset x_L^{k+1} \subset \dots, \quad 2 \leq k < \infty$$

characterized by increasing finite multiplicities.

2.2 The versal deformation

2.2.1 Generalities on the versal deformation

The versal deformation will be considered for germs of differentiable functions $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta}})$) of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$). These left (resp. right) differentiable functions will be written $\phi_{j_\delta L}(x_L)$ (resp. $\phi_{j_\delta R}(x_R)$) where x_L (resp. x_R) is a n -tuple of numbers

$$\begin{aligned} x_L &= (x_{1_L}, x_{2_L}, \dots, x_{n_L}) \in (F_L^+)^n \\ (\text{resp. } x_R &= (x_{1_R}, x_{2_R}, \dots, x_{n_R}) \in (F_R^+)^n). \end{aligned}$$

So, on the j_δ -th conjugacy class of the algebraic semigroup $G^{(n)}(F_v^+) \equiv T_n(F_v^+)$ (resp. $G^{(n)}(F_v^+) \equiv T_n^t(F_v^+)$), there is:

- 1) a set of left (resp. right) differentiable functions $\phi_{j_\delta, m_{j_\delta}}(x_L)$ (resp. $\phi_{j_\delta, m_{j_\delta}}(x_R)$) which are the sections of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$).
- 2) a germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) (or a set of germs) of differentiable functions, where ω_L (resp. ω_R) denotes a m -tuple of numbers, $1 \leq m \leq 3$.

This germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) is assumed to be:

- simple and of corank ≤ 3 ;
- associated with an isolated singularity of order k .

As the corank of the singularity is not superior to 3, the n -tuple of numbers of $(F_L^+)^n$ (resp. $(F_R^+)^n$), restricted to a small domain centered on the singularity, will be rewritten according to:

$$\begin{aligned} x'_L &= (x_{1_L}, \dots, x_{n_L - m_L}, \omega_{1_L}, \dots, \omega_{m_L}), & 1 \leq m_L \leq 3 \\ \text{(resp. } x'_R &= (x_{1_R}, \dots, x_{n_R - m_R}, \omega_{1_R}, \dots, \omega_{m_R}) \text{).} \end{aligned}$$

The finite determinacy was first envisaged for germs having an isolated singularity: this is the pioneer work of H. Grauert and H. Kerner [G-K], R. Thom, [Tho1], [Tho2], [Lev], J. Mather, [Mat1], [Mat2], [Mat3], V.I. Arnold [Arn1], J.C. Tougeron [Tou], B. Malgrange [Mal], and others.

Afterwards, this problem was generalized to functions having a fixed analytic set Σ as critical set. If I denotes the ideal of functions on this critical set Σ , the finite I -determinacy of these functions and their versal I -unfoldings were considered and proved in [Sie] and [Pel].

With this in view, the preparation theorem and the versal deformation will be recalled for germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of differentiable functions having an isolated singularity of corank 1 and order k .

2.2.2 The division theorem

Let $x'_L = (x_{1_L}, \dots, x_{n_L - 1}, \omega_L)$ (resp. $x'_R = (x_{1_R}, \dots, x_{n_R - 1}, \omega_R)$) denote the coordinates of $(F_L^+)^n$ (resp. $(F_R^+)^n$).

A germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) has a singularity of corank 1 (then, $m = 1$) and order k in ω_L (resp. ω_R) if $\phi_{j_\delta}(0, \omega_L) = \omega_L^k e_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(0, \omega_R) = \omega_R^k e_{j_\delta}(\omega_R)$), where $e_{j_\delta}(\omega_L)$ (resp. $e_{j_\delta}(\omega_R)$) is a differentiable unit, i.e. verifying $e_{j_\delta}(0) \neq 0$ (resp. $e_{j_\delta}(0) \neq 0$).

Let $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) be the algebra of polynomials in ω_L (resp. ω_R) with coefficients $a_{ij_\delta}(x_L)$ (resp. $a_{ij_\delta}(x_R)$), being ideals of functions defined on a domain $D_L \subset B_L$ (resp. $D_R \subset B_R$) where:

- B_L (resp. B_R) is an upper (resp. lower) half open ball centered on ω_L (resp. ω_R) in $\phi_{j_\delta}(x_L)$ (resp. $\phi_{j_\delta}(x_R)$);
- $x_L = (x_{1_L}, \dots, x_{n_L-1})$ (resp. $x_R = (x_{1_R}, \dots, x_{n_R-1})$) is the $(n-1)$ -tuple of x'_L (resp. x'_R) in $(F_L^+)^{n-1}$ (resp. $(F_R^+)^{n-1}$).

If the germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) has order k in ω_L (resp. ω_R), then, for every n -dimensional differentiable function (germ) $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$), there exists a $(n-1)$ -dimensional differentiable function (germ) $q_{j_{\delta_L}}$ (resp. $q_{j_{\delta_R}}$) and a polynomial:

$$R_{j_{\delta_L}} = \sum_{i=1}^s a_{ij_\delta}(x_L) \omega_{j_{\delta_L}}^i \in \theta[\omega_L]$$

(resp. $R_{j_{\delta_R}} = \sum_{i=1}^s a_{ij_\delta}(x_R) \omega_{j_{\delta_R}}^i \in \theta[\omega_R]$)

with degree $s < k$ such that:

$$f_{j_{\delta_L}} = \phi_{j_{\delta_L}}(\omega_L) \cdot q_{j_{\delta_L}} + R_{j_{\delta_L}}$$

(resp. $f_{j_{\delta_R}} = \phi_{j_{\delta_R}}(\omega_R) \cdot q_{j_{\delta_R}} + R_{j_{\delta_R}}$)

be the division theorem (adapted to the left and right cases) introduced by B. Malgrange in [Mal]. The Malgrange division theorem, closely related to the version of J. Mather [Mat1], [Mat2], [Mat3], is the differentiable version of the Weierstrass division theorem [G-R].

2.2.3 The division theorem (corank 2 case)

The division theorem, recalled in section 2.2.2 for germs of differentiable functions having an isolated singularity of corank 1, can easily be generalized to germs $\phi_{j_\delta}(\omega_{1_L}, \omega_{2_L})$ (resp. $\phi_{j_\delta}(\omega_{1_R}, \omega_{2_R})$) having an isolated singularity of corank 2.

Indeed, a germ $\phi_{j_\delta}(\omega_{1_L}, \omega_{2_L})$ (resp. $\phi_{j_\delta}(\omega_{1_R}, \omega_{2_R})$) has a singularity of corank 2 and order k in ω_{1_L} and ω_{2_L} (resp. ω_{1_R} and ω_{2_R}) if:

$$\text{a) } \phi_{j_\delta}(\omega_{1_L}, \omega_{2_L}) = P_{j_\delta}(\omega_{1_L}, \omega_{2_L}) e_{j_\delta}(\omega_{1_L}, \omega_{2_L})$$

(resp. $\phi_{j_\delta}(\omega_{1_R}, \omega_{2_R}) = P_{j_\delta}(\omega_{1_R}, \omega_{2_R}) e_{j_\delta}(\omega_{1_R}, \omega_{2_R})$)

where:

- $P_{j_\delta}(\omega_{1_L}, \omega_{2_L})$ (resp. $P_{j_\delta}(\omega_{1_R}, \omega_{2_R})$) is a polynomial of degree k in ω_{1_L} or in ω_{2_L} (resp. ω_{1_R} or in ω_{2_R});

- $e_{j_\delta}(\omega_{1_L}, \omega_{2_L})$ (resp. $e_{j_\delta}(\omega_{1_R}, \omega_{2_R})$) is a differentiable unit.

b) the polynomial $R_{j_{\delta_L}} \in \theta[\omega_{1_L}, \omega_{2_L}]$ (resp. $R_{j_{\delta_R}} \in \theta[\omega_{1_R}, \omega_{2_R}]$) of the quotient algebra has degree $\ell < k$, in such a way that this quotient algebra be a finitely generated tensorial space of type $(0, 2)$ and dimension $\ell \leq k$.

2.2.4 singularization of the semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$)

Let $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) be the semisheaf of left (resp. right) smooth differentiable functions $\phi_{j_\delta}(x_L)$ (resp. $\phi_{j_\delta}(x_R)$).

And, let

$$\begin{aligned} \bar{\rho}_{G_L} : \theta_{G_L^{(n)}} &\longrightarrow \theta_{G_L^{(n)}}^* \\ (\text{resp. } \bar{\rho}_{G_R} : \theta_{G_R^{(n)}} &\longrightarrow \theta_{G_R^{(n)}}^*) \end{aligned}$$

be the singularization of $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$), in the sense of proposition 2.1.6, in such a way that $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) be the semisheaf whose sections $\phi_{G_{j_\delta L}}^{*(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_\delta R}}^{*(n)}(x_{g_{j_\delta}})$) are the differentiable functions $\phi_{G_{j_\delta L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{G_{j_\delta R}}^{(n)}(x_{g_{j_\delta}})$) endowed with germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) having degenerate singularities of corank 1.

2.2.5 Proposition (Versal deformation)

The versal deformation of the semisheaf $\theta_{G_L^{(n)}}^$ (resp. $\theta_{G_R^{(n)}}^*$) is given by the contracting fibre bundle:*

$$\begin{aligned} D_{S_L} : (\theta_{G_L^{(n)}}^* \setminus \theta_L(a)) \times \theta[\omega_L] &\longrightarrow \theta_{G_L^{(n)}}^* \\ (\text{resp. } D_{S_R} : (\theta_{G_R^{(n)}}^* \setminus \theta_R(a)) \times \theta[\omega_R] &\longrightarrow \theta_{G_R^{(n)}}^*) \end{aligned}$$

where:

- $\theta_L(a)$ (resp. $\theta_R(a)$) is the (semi)sheaf of ideals $a_{ij_\delta}(x_L)$ (resp. $a_{ij_\delta}(x_R)$) of differentiable functions as introduced in section 2.2.2.
- $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) is the algebra of polynomials $R_{j_{\delta_L}}$ (resp. $R_{j_{\delta_R}}$) introduced in section 2.2.2

and whose fibre

$$\begin{aligned} \theta_{S_L} &= \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_L^i), \dots, \theta^1(\omega_L^s)\} \\ (\text{resp. } \theta_{S_R} &= \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}) \end{aligned}$$

is the family of the (semi-)sheaves of the left (resp. right) base S_L (resp. S_R) of the versal deformation.

Proof. θ_{f_L} (resp. θ_{f_R}) is the (semi)sheaf of differentiable functions

$$\begin{aligned} f_{j_{\delta_L}} &= \phi_{j_{\delta}}(\omega_L) \cdot q_{j_{\delta_L}} + R_{j_{\delta_L}} \\ (\text{resp. } f_{j_{\delta_R}} &= \phi_{j_{\delta}}(\omega_R) \cdot q_{j_{\delta_R}} + R_{j_{\delta_R}}) \end{aligned}$$

introduced in section 2.2.2.

The polynomials $R_{j_{\delta_L}} \in \theta[\omega_L]$ (resp. $R_{j_{\delta_R}} \in \theta[\omega_R]$) have as coefficients the ideals of functions $a_{ij_{\delta}}(x_L)$ (resp. $a_{ij_{\delta}}(x_R)$) on the differentiable functions $\phi_{j_{\delta}}(x_L)$ (resp. $\phi_{j_{\delta}}(x_R)$): so we have

$$\begin{aligned} a_{j_{\delta_L}}(x_L) &\subset \phi_{j_{\delta}}(x_L) \\ (\text{resp. } a_{j_{\delta_R}}(x_R) &\subset \phi_{j_{\delta}}(x_R)). \end{aligned}$$

Then, it appears that:

$$\begin{aligned} \theta_{G_L^{(n)}}^* \times \theta_{S_L} &= (\theta_{G_L^{(n)}}^* \setminus \theta_L(a)) \times \theta[\omega_L] \\ (\text{resp. } \theta_{G_R^{(n)}}^* \times \theta_{S_R} &= (\theta_{G_R^{(n)}}^* \setminus \theta_R(a)) \times \theta[\omega_R]), \end{aligned}$$

and, thus, that θ_{S_L} (resp. θ_{S_R}) is the fibre of the contracting fibre bundle D_{S_L} (resp. D_{S_R}) rewritten as follows [Ste]:

$$\begin{aligned} D_{S_L} : \quad \theta_{G_L^{(n)}}^* \times \theta_{S_L} &\longrightarrow \theta_{G_L^{(n)}}^* \\ (\text{resp. } D_{S_R} : \quad \theta_{G_R^{(n)}}^* \times \theta_{S_R} &\longrightarrow \theta_{G_R^{(n)}}^*). \end{aligned} \quad \blacksquare$$

2.2.6 Proposition

Let $\theta_{G_L^{(n)}}^{\text{vers}} = \theta_{G_L^{(n)}}^* \times \theta_{S_L}$ (resp. $\theta_{G_R^{(n)}}^{\text{vers}} = \theta_{G_R^{(n)}}^* \times \theta_{S_R}$) denote the semisheaf unfolded from $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$): it is the total space of the fibre bundle D_{S_L} (resp. D_{S_R}).

Let $\Sigma_{G_L^{(n)}}^*$ (resp. $\Sigma_{G_R^{(n)}}^*$) be the singular locus of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$): it is the sheaf $\theta_{\phi_{\omega_L}}$ (resp. $\theta_{\phi_{\omega_R}}$) of germs $\phi_{j_{\delta}}(\omega_L)$ (resp. $\phi_{j_{\delta}}(\omega_R)$) of differentiable functions.

Then, we have that:

- a) the unfolded image $D_{S_L}^{-1}(\theta_{\phi_{\omega_L}})$ (resp. $D_{S_R}^{-1}(\theta_{\phi_{\omega_R}})$) of the singular locus $\Sigma_{G_L^{(n)}}^*$ (resp. $\Sigma_{G_R^{(n)}}^*$) is the (semi)sheaf θ_{f_L} (resp. θ_{f_R}):

$$\begin{aligned} \theta_{f_L} &= D_{S_L}^{-1}(\theta_{\phi_{\omega_L}}) \\ (\text{resp. } \theta_{f_R} &= D_{S_R}^{-1}(\theta_{\phi_{\omega_R}})); \end{aligned}$$

$$\begin{aligned}
b) \quad D_{S_L}^{-1}(\theta_{G_L^{(n)}}^* \setminus \theta_{\phi_{\omega_L}}) &= (\theta_{G_L^{(n)}}^* \setminus D_{S_L}^{-1}(\theta_{\phi_{\omega_L}})) \setminus \theta_{S_L} \\
(\text{resp. } D_{S_R}^{-1}(\theta_{G_R^{(n)}}^* \setminus \theta_{\phi_{\omega_R}}) &= (\theta_{G_R^{(n)}}^* \setminus D_{S_R}^{-1}(\theta_{\phi_{\omega_R}})) \setminus \theta_{S_R}).
\end{aligned}$$

Proof. By versal deformation, θ_{S_L} (resp. θ_{S_R}) is the fibre of the sheaf $\theta_{\phi_{\omega_L}}$ (resp. $\theta_{\phi_{\omega_R}}$) of germs of differentiable functions, outside of which D_{S_L} (resp. D_{S_R}) is an isomorphism: this is reflected by the equality b) of this proposition.

2.2.7 Definition

The quotient algebra $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) of germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) having a singularity of corank 1 and multiplicity $(k-1)$ is the quotient of the algebra \mathcal{E}_{n_L} (resp. \mathcal{E}_{n_R}) of function germs (generally, it is the algebra of integer power series) by the graded ideal $I_{\phi_{\omega_L}}$ (resp. $I_{\phi_{\omega_R}}$) of $\phi_{j_L}(\omega_L)$ (resp. $\phi_{j_R}(\omega_R)$):

$$\begin{aligned}
\theta[\omega_L] &= \mathcal{E}_{n_L} / I_{\phi_{\omega_L}}, \quad n = 1 \\
(\text{resp. } \theta[\omega_R] &= \mathcal{E}_{n_R} / I_{\phi_{\omega_R}})
\end{aligned}$$

where:

$$\begin{aligned}
I_{\phi_{\omega_L}} &= \mathcal{E}_{n_L} \langle \phi_L^{(1)}, \dots, \phi_L^{(k-1)} \rangle \\
(\text{resp. } I_{\phi_{\omega_R}} &= \mathcal{E}_{n_R} \langle \phi_R^{(1)}, \dots, \phi_R^{(k-1)} \rangle)
\end{aligned}$$

is generated by the partial derivatives $\phi_L^{(k-1)}$ (resp. $\phi_R^{(k-1)}$) of $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$), [A-G-L-V].

The quotient algebra is thus finitely generated: it is composed of the polynomials $R_{j_{\delta_L}} \in \theta[\omega_L]$ (resp. $R_{j_{\delta_R}} \in \theta[\omega_R]$) (see section 2.2.2), which generate vector (semi)spaces of dimension $s < k$, and proceeds from a set of contracting morphisms extended those considered in the singularization processes as developed in proposition 2.1.6.

2.2.8 Proposition

1) The versal deformations of the germ $y_L = \omega_L^{k+1}$ (resp. $y_R = \omega_R^{k+1}$) of differentiable functions $\phi_{G_{j_{\delta_L}}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_{\delta_R}}}^{(n)}(x_{g_{j_{\delta}}})$) is generated by a sequence

$$\begin{aligned}
D_{S_L^{(k+1)}} &= (D_{S_L^{(k+1)}}^{(1)}, \dots, D_{S_L^{(k+1)}}^{(i)}, \dots, D_{S_L^{(k+1)}}^{(k-1)}) \\
(\text{resp. } D_{S_R^{(k+1)}} &= (D_{S_R^{(k+1)}}^{(1)}, \dots, D_{S_R^{(k+1)}}^{(i)}, \dots, D_{S_R^{(k+1)}}^{(k-1)})
\end{aligned}$$

of $(k-2)$ contracting morphisms $D_{S_L^{(k+1)}}^{(i)}$ (resp. $D_{S_R^{(k+1)}}^{(i)}$) extending the sequence of contracting surjective morphisms $\bar{p}_L^{(\text{sing})}$ (resp. $\bar{p}_R^{(\text{sing})}$) of singularization according to the following diagram:

$$\begin{array}{ccccccc}
\bar{p}_L^{(\text{sing})} : D_L^{(0)} \rightarrow \dots & & & & & & \\
\begin{array}{ccccccc}
\bar{p}_L^{(3)} & \xrightarrow{\quad} & \omega_L^{(3)} \cup D_L^{(3)} & \rightarrow \dots \xrightarrow{\quad} & \bar{p}_L^{(i)} & \xrightarrow{\quad} & \omega_L^{(i)} \cup D_L^{(i)} \rightarrow \dots \xrightarrow{\quad} & \bar{p}_L^{(k)} & \xrightarrow{\quad} & \omega_L^{(k)} \cup D_L^{(k)} & \xrightarrow{\quad} & \bar{p}_L^{(k+1)} & \xrightarrow{\quad} & \omega_L^{(k+1)} \\
& & \cup & & \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\
& & D_L'^{(1)} & & D_L'^{(i-2)} & & D_L'^{(k-2)} & & D_L'^{(k-1)} & & D_L'^{(k-1)} & & D_L'^{(k-1)} & & D_L'^{(k-1)} \\
& & \downarrow D_{S_L^{(k+1)}}^{(1)} & & \downarrow D_{S_L^{(k+1)}}^{(i-2)} & & \downarrow D_{S_L^{(k+1)}}^{(k-2)} & & \downarrow D_{S_L^{(k+1)}}^{(k-1)} & & \downarrow D_{S_L^{(k+1)}}^{(k-1)} & & \downarrow D_{S_L^{(k+1)}}^{(k-1)} & & \downarrow D_{S_L^{(k+1)}}^{(k-1)} \\
& & \omega_L^3 & \longrightarrow & \omega_L^i & \longrightarrow & \omega_L^k & \longrightarrow & \omega_L^{k+1} & & \omega_L^{k+1} & & \omega_L^{k+1} & & \omega_L^{k+1} \\
& & + a_1 \omega_L^1 & & + \sum_{i=1}^{i-2} a_i \omega_L^i & & + \sum_{i=1}^{k-2} a_i \omega_L^i & & + \sum_{i=1}^{k-1} a_i \omega_L^i & & + \sum_{i=1}^{k-1} a_i \omega_L^i & & + \sum_{i=1}^{k-1} a_i \omega_L^i & & + \sum_{i=1}^{k-1} a_i \omega_L^i
\end{array}
\end{array}$$

(idem for the R -case)

where $D_L'^{(i-2)}$ is a normal crossings divisor on a real (or a set of) irreducible completion(s) $F_{v_{j\delta}^{j\delta}}^+$ mapped onto the neighbourhood of the germ $y_L = \omega_L^i$.

2) If $\phi_{G_{j\delta_L}}^{*(n,i)}(x_{g_{j\delta}})$ denotes the function having a germ $y_L = \omega_L^i$ of codimension $(i-2)$, then, the $D_{S_L^{(k+1)}}^{(i-2)}$ contracting morphism corresponds to the contracting fibre bundle:

$$D_{S_L^{(k+1)}}^{(i-2)} : \phi_{G_{j\delta_L}}^{*(n,i)}(x_{g_{j\delta}}) \times \omega_L^{i-2} \longrightarrow \phi_{G_{j\delta_L}}^{*(n,i)}(x_{g_{j\delta}})$$

in such a way that:

- $D_{S_L^{(k+1)}}^{(i-2)} \subset D_{S_L^{(k+1)}}^{(i-1)}$;
- ω_L^{i-2} is the contracting fibre, i.e. the divisor $D_L'^{(i-2)}$ in the neighbourhood of $y_L = \omega_L^i$ on $a_{(i-2)j}(x_L) \subset \phi_{G_{j\delta_L}}^{*(n,i)}(x_{g_{j\delta}})$.

Proof.

1) The versal deformation of a germ $y_L = \omega_L^{k+1}$ is an extension of its singularization as described in proposition 2.1.10. Indeed, to the i -th contracting surjective morphism of singularization:

$$\bar{\rho}_L^{(i)} : \omega_L^{i-1} \cup D_L^{(i-1)} \longrightarrow \omega_L^i, \quad 1 \leq i \leq k+1,$$

introduced in proposition 2.1.10 as a contracting fibre bundle whose fibre $D_L^{(i-1)}$ (which is a normal crossings divisor) collapses into one point on the germ $y_L = \omega_L^{i-1}$, corresponds the contracting fibre bundle $D_{S_L^{(k+1)}}^{(i-2)}$ of the versal deformation $D_{S_L^{(k+1)}}$:

$$D_{S_L^{(k+1)}}^{(i-2)} : \phi_{G_{j_L}}^{*(n,i)}(x_{g_{j_L}}) \times \omega_L^{i-2} \longrightarrow \phi_{G_{j_L}}^{*(n,i)}(x_{g_{j_L}})$$

in such a way that the divisor $D_L'^{(i-2)}$ be projected in the neighbourhood of the singular germ $y_L = \omega_L^{k+1}$ where it is rewritten ω_L^{i-2} .

So, from the third contracting surjective morphism of singularization $\bar{\rho}_L^{(3)}$, where the singularity becomes degenerated, we can associate to each contracting surjective morphism of singularization $\bar{\rho}_L^{(i)}$, $3 \leq i \leq k+1$, a contracting fibre bundle $D_{S_L^{(k+1)}}^{(i-2)}$, $i-2 \leq i \leq k+1$, of versal deformation.

2) And, to the sequence of singularizations:

$$\bar{\rho}_L^{(3)}, \dots, \bar{\rho}_L^{(i)}, \dots, \bar{\rho}_L^{(k)}, \bar{\rho}_L^{(k+1)},$$

corresponds the sequence of versal subdeformations:

$$D_{S_L^{(k+1)}}^{(1)} \subset \dots \subset D_{S_L^{(k+1)}}^{(i-2)} \subset \dots \subset D_{S_L^{(k+1)}}^{(k-2)} \subset D_{S_L^{(k+1)}}^{(k-1)},$$

which are embedded and which correspond to the sequence of $(k-1)$ embedded vector sub(semi)spaces generated by the polynomials:

$$R_{j_{\delta_L}}^{(1)} \subset \dots \subset R_{j_{\delta_L}}^{(i-2)} \subset \dots \subset R_{j_{\delta_L}}^{(k-2)} \subset R_{j_{\delta_L}}^{(k-1)},$$

where $R_{j_{\delta_L}}^{(i-2)}$ is the truncated polynomial of the quotient algebra $\theta[\omega_L]$ introduced in section 2.2.2 and given by:

$$R_{j_{\delta_L}}^{(i-2)} = \sum_{i=1}^{i-2} a_{ij_{\delta}}(x_L) \omega_{j_{\delta_L}}^i.$$

3) The order of the divisors $D_L'^{(i-2)}$, projected in the neighbourhood of the singular germ $y_L = \omega_L^{k+1}$, increases in function of the increase of the dimension of the generated vector sub(semi)spaces $R_j^{(i-2)}$, $i-2 \leq i \leq k+1$, of the versal deformation, because the space around the singularity becomes “over” compact when the dimension of the versal deformation increases. In fact, it will be proved in the following that the geometry of the space around the singularity deformed by versal unfolding is spherical.

■

2.2.9 Corollary

The sequence $D_{S_L^{(k+1)}}$ (resp. $D_{S_R^{(k+1)}}$) of contracting morphisms generating the versal deformations of singular germs of corank 1 and multiplicity ≥ 1 explains why the quotient algebra $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) of the versal deformation is finitely determined [Mat3].

2.3 The geometry of the versal deformation

The geometry of the versal deformation will be envisaged for the sections $\phi_{G_{jL}}^{(n)}(x_{g_{j\delta}})$ (resp. $\phi_{G_{jR}}^{(n)}(x_{g_{j\delta}})$) of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}^{(n)}$ (resp. $\theta_{G_R^{(n)}}^{(n)}$). These sections are assumed to be differentiable functions having singular germs $\phi_{j\delta}(\omega_L)$ (resp. $\phi_{j\delta}(\omega_R)$) of corank $m \leq 3$ and multiplicity “ i ”, $1 \leq i \leq n$ (see section 2.2.1).

Let $\Sigma_{\phi_{G_{jL}}^{(n)}}$ (resp. $\Sigma_{\phi_{G_{jR}}^{(n)}}$) denote the singular locus of a singular germ $\phi_{j\delta}(\omega_L)$ (resp. $\phi_{j\delta}(\omega_R)$) of corank m and multiplicity “ i ” and let $D_{\Sigma_{\phi_{G_{jL}}^{(n)}}}$ (resp. $D_{\Sigma_{\phi_{G_{jR}}^{(n)}}}$) be the neighbourhood of this singular locus whose curvature is affected by the singularity.

2.3.1 Proposition

The geometry is hyperbolic on the neighbourhood $D_{\Sigma_{\phi_{G_{jL}}^{(n)}}}$ (resp. $D_{\Sigma_{\phi_{G_{jR}}^{(n)}}}$) of the singular locus $\Sigma_{\phi_{G_{jL}}^{(n)}}$ (resp. $\Sigma_{\phi_{G_{jR}}^{(n)}}$) of a not unfolded singular germ of corank $m \leq 3$ and multiplicity “ i ”, $1 \leq i \leq n$.

Proof. The main idea consists in showing that there is a deviation to euclidicity in $D_{\Sigma_{\phi_{G_{jL}}^{(n)}}}$ (resp. $D_{\Sigma_{\phi_{G_{jR}}^{(n)}}}$) of the differentiable function $\phi_{G_{jL}}^{(n)}(x_{g_{j\delta}})$ (resp. $\phi_{G_{jR}}^{(n)}(x_{g_{j\delta}})$) of dimension n .

This deviation to euclidicity can be evaluated by searching the conditions to which must satisfy the metric $ds^2 = g_{ij} du^i du^j$ in the neighbourhood $D_{\Sigma_{\phi_{G_{jL}}^{(n)}}}$ (resp. $D_{\Sigma_{\phi_{G_{jR}}^{(n)}}}$) of the singular locus.

General conditions in the Euclidean and non Euclidean cases were developed by E. Cartan in his classical book “leçons sur la géométrie des espaces de Riemann” [Car] to which we refer.

The developments will be envisaged for the left and right cases without distinction: thus, we omit the “ L ” and “ R ” and we simplify the notations:

- $D_{\Sigma_{\phi_{G_{jL}}^{(n)}}}$ and $D_{\Sigma_{\phi_{G_{jR}}^{(n)}}}$ become D_{Σ} ;
- $\Sigma_{\phi_{G_{jL}}^{(n)}}$ and $\Sigma_{\phi_{G_{jR}}^{(n)}}$ become Σ ;
- $\phi_{G_{jL}}^{(n)}(x_{g_{j\delta}})$ and $\phi_{G_{jR}}^{(n)}(x_{g_{j\delta}})$ become $\phi^{(n)}$.

- 1) Consider first the Euclidean case and remark that there does not exist in general a coordinate system giving to the Euclidean space a fix metric.

Let M be a point of coordinates (u^1, \dots, u^n) on $\phi^{(n)} \setminus D_{\Sigma} \setminus \Sigma$ and let $(\vec{e}_1, \dots, \vec{e}_n)$ be the basis vectors of the proper referential centred on the point M , such that the

components:

- $dM = du^i \vec{e}_i$, $1 \leq i \leq n$,
- $d\vec{e}_i = \Gamma_{ij}^k du^j \vec{e}_k$ where $\Gamma_{ij}^k = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right)$,

do not belong to a stratum on $(D_\Sigma \cup \Sigma)$.

The integrability conditions of $d\vec{e}_i = \Gamma_{ij}^k du^j \vec{e}_k$ are precisely the searched condition to which the g_{ij} must satisfy.

These integrability conditions can be obtained geometrically as follows.

To each point M will correspond a point P in the neighbourhood of M . This point P is defined by its coordinates (x^1, \dots, x^n) with respect to the referential of the point M .

The components of the differential of the point P are given by:

$$Dx^i = dx^i + du^i + x^k \Gamma_{kr}^i du^r$$

such that

$$D_r x^i = \frac{\partial x^i}{\partial u^r} + \delta_r^i + x^k \Gamma_{kr}^i \quad \text{with} \quad \delta_r^i \begin{cases} = 0 , & \text{if } i \neq r , \\ = 1 , & \text{if } i = r . \end{cases}$$

Let then M' , M'' and M''' be the points obtained as follows: the first M' is obtained by increasing the coordinate u^r by an infinitesimal parameter α , the second M'' by increasing the coordinate u^s by an infinitesimal parameter β , and the third M''' by increasing the coordinate u^r by α and the coordinate u^s by β .

Let P' , P'' and P''' be the points corresponding to the points M' , M'' and M''' respectively.

The infinitesimal small vector $\overrightarrow{PP'}$ has the “ i ” contravariant components given by $\alpha D_r x^i$.

On the other hand, to the elementary variations $\overline{MM'''} = \{\delta u^1 = 0, \dots, \delta u^r = \alpha, \dots, \delta u^s = \beta, \dots, \delta u^n = 0\}$ of the points M will correspond the infinitesimal small vector $\overrightarrow{P''P'''} - \overrightarrow{PP'}$ whose components $\alpha\beta D_s D_r x^i$ are given by:

$$\begin{aligned} D_s D_r x^i &= \frac{\partial D_r x^i}{\partial u^s} + D_r x^k \Gamma_{ks}^i \\ &= \frac{\partial^2 x^i}{\partial u^r \partial u^s} + \frac{\partial x^k}{\partial u^s} \Gamma_{kr}^i + x^k \frac{\partial \Gamma_{kr}^i}{\partial u^s} + \frac{\partial x^k}{\partial u^r} \Gamma_{ks}^i + \Gamma_{rs}^i + x^k \Gamma_{kr}^h \Gamma_{hs}^i . \end{aligned}$$

Similarly, the infinitesimal small vector $\overrightarrow{P'P''} - \overrightarrow{PP''}$ has for components $\alpha\beta D_r D_s x^i$. An elementary calculus gives that $D_r D_s x^i - D_s D_r x^i = 0$, which leads to the searched integrability conditions of $d\vec{e}_i = \Gamma_{ij}^k du^j \vec{e}_k$:

$$\frac{\partial \Gamma_i^{kr}}{\partial u^s} - \frac{\partial \Gamma_i^{ks}}{\partial u^r} + (\Gamma_{ir}^h \Gamma_{hs}^k - \Gamma_{is}^h \Gamma_{hr}^k) = 0$$

corresponding to the conditions to which the Euclidean metric g_{ij} must satisfy.

- 2) Consider the integrability conditions of $d\vec{e}_{i'}$ for the components i' , $1 \leq i' \leq m$, of a stratum Σ of corank m .

Each function $\phi^{(n)}(u_1, \dots, u_i, \dots, u_{n-m}, \dots, v_1, \dots, v_{i'}, \dots, v_m)$, $1 \leq i \leq n-m$, $1 \leq i' \leq m$, having M as critical point(s) [Tho1] on Σ must satisfy:

$$\left. \frac{\partial \phi^{(n)}}{\partial v_1} \right|_M = \dots = \left. \frac{\partial \phi^{(n)}}{\partial v_m} \right|_M = 0,$$

i.e.

$$\lim_{\Delta v_{i'} \rightarrow 0} \left(\phi^{(n)}(u_1, \dots, u_i, \dots, u_{n-m}, v_1, \dots, v_{i'} + \Delta v_{i'}, \dots, v_m) - \phi^{(n)}(u_1, \dots, u_i, \dots, u_{n-m}, v_1, \dots, v_{i'}, \dots, v_m) \right) / \Delta v_{i'} \Big|_M = 0, \quad 1 \leq i' \leq m,$$

which implies that:

$$\begin{aligned} \lim_{\Delta v_{i'} \rightarrow 0} \phi^{(n)}(u_1, \dots, u_{n-m}, v_1, \dots, v_{i'} + \Delta v_{i'}, \dots, v_m) \\ = \phi^{(n)}(u_1, \dots, u_{n-m}, v_1, \dots, v_{i'}, \dots, v_m) \end{aligned}$$

in the neighbourhood D_Σ of Σ .

This means that, in the $1 \leq i' \leq m$ dimensions of the singular locus Σ , the differentials of the basic vectors $\vec{e}_{i'}$ must be given by:

$$d\vec{e}_{i'} = \Gamma_{i'j'}^{k'} dv^{j'} \vec{e}_{k'} - \kappa g_{i'k'} dv^{k'} \cdot M(u^1, \dots, u^{n-m}, v^1, \dots, v^m), \quad \text{with } \kappa \in \mathbb{R}^+,$$

because $\|d\vec{e}_{i'}\| < \|d\vec{e}_i\|$ in the neighbourhood D_Σ of the singular locus Σ .

Similarly, the components of the differential of the point $P(x^1, \dots, x^{n'})$ in the i' , $1 \leq i' \leq m$, dimensions of Σ will then be:

$$Dx^{i'} = dx^{i'} + dv^{i'} + x^{k'} \Gamma_{k'r'}^{i'} dv^{r'} - \kappa g_{i'k'} dv^{k'}$$

such that

$$D_{r'} x^{i'} = \frac{\partial x^{i'}}{\partial v^{r'}} + \delta_{r'}^{i'} + x^{k'} \Gamma_{k'r'}^{i'} - \kappa g_{i'k'} \delta_{r'}^{k'}$$

corresponds to the r' -th component of the infinitesimally small vector $\overrightarrow{PP'}/\alpha'$.
 Calculating $D_{r'}D_{s'}x^{i'} - D_{s'}D_{r'}x^{i'}$, we find the integrability conditions of $d\vec{e}_{i'}$:

$$\frac{\partial \Gamma_{i'r'}^{k'}}{\partial v^{s'}} - \frac{\partial \Gamma_{i's'}^{k'}}{\partial v^{r'}} + (\Gamma_{i'r'}^{h'}\Gamma_{h's'}^{k'} - \Gamma_{i's'}^{h'}\Gamma_{h'r'}^{k'}) = -\kappa (\delta_{s'}^{k'}g_{i'r'} - \delta_{r'}^{k'}g_{i's'})$$

which clearly do not correspond to the conditions given in 1) to which the coefficients g_{ij} must satisfy in order that ds^2 be Euclidean.

We thus have a non-Euclidean hyperbolic metric of curvature “ $-\kappa$ ” on each stratum D_Σ in the neighbourhood of the singular locus Σ .

- 3) These developments correspond to those of Hironaka [Hir3] who showed that there is a normal cone along the singular locus Σ . ■

2.3.2 The limit set of the Kleinian group

A Kleinian group G of $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is the group of Möbius transformations of $\overline{\mathbb{R}}^n$ if it acts discontinuously somewhere in \mathbb{R}^n .

The action of the Kleinian group G of $\overline{\mathbb{R}}^n$ can be extended to $\overline{H}^{n+1} = H^{n+1} \cup \overline{\mathbb{R}}^n$ where $H^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ is the $(n+1)$ -dimensional hyperbolic space: G thus acts as a group of isometries of H^{n+1} with the hyperbolic metric.

The orbit space M_G associated with the Kleinian group G is defined by:

$$M_G = \overline{H}^{n+1} \setminus L(G))/G$$

where $L(G)$ denotes the limit set of a discrete Kleinian group G [Mil1], [Tuk1], [Tuk2].

This limit set is the closure of the set of fixed points of non-elliptic elements of G [Abi]. It is a nowhere dense set whose area measure is zero: this corresponds to the zero-measure problem of Ahlfors [Ahl].

A discrete Kleinian group G is said to be elementary if $L(G)$ consists of at most two points.

An ordinary set $\Omega(G)$ of a Kleinian group G is defined by $\Omega(G) = \overline{\mathbb{R}}^n \setminus L(G)$: it is the region of discontinuity of G .

Recall that a Möbius transformation g of $\overline{\mathbb{R}}^n$ is loxodromic if it is a transformation of the form $g(x) = \lambda\alpha(x)$ where $x \in \mathbb{R}^n$, $\lambda > 1$ and $\alpha \in O(n)$ is the orthogonal group of \mathbb{R}^n . g is hyperbolic if $\alpha = \text{id.}$, elliptic if $\lambda = 1$ and parabolic if g has the form $g(x) = \alpha(x) + a$ where $a \in \mathbb{R}^n \setminus \{0\}$ and $\alpha(a) = a$.

2.3.3 Left and right actions of the Kleinian group

Similarly, we can introduce left (resp. right) Möbius transformations g_L (resp. g_R) acting discontinuously in $(F_L^+)^n$ (resp. $(F_R^+)^n$) in the upper (resp. lower) half space and left (resp. right) actions of the Kleinian group on the upper (resp. lower) n -dimensional hyperbolic half space H_L^n (resp. H_R^n).

The left (resp. right) orbit space M_{G_L} (resp. M_{G_R}) associated with the left (resp. right) action of the Kleinian group is given by:

$$M_{G_L} = H_L^n \setminus L(G_L)/G_L \quad (\text{resp. } M_{G_R} = H_R^n \setminus L(G_R)/G_R)$$

where $L(G_L)$ (resp. $L(G_R)$) denotes the limit set of the Kleinian group G_L (resp. G_R) acting on the upper (resp. lower) half space.

And, a left (resp. right) ordinary set $\Omega(G_L)$ (resp. $\Omega(G_R)$) of G_L (resp. G_R) is defined by:

$$\Omega_{G_L} = (\overline{F}_L^+)^n \setminus L(G_L) \quad (\text{resp. } \Omega_{G_R} = (\overline{F}_R^+)^n \setminus L(G_R))$$

where

$$(\overline{F}_L^+)^n = (F_L^+)^n \cup \{\infty\} \quad (\text{resp. } (\overline{F}_R^+)^n = (F_R^+)^n \cup \{\infty\}).$$

2.3.4 Proposition

Let $\Sigma_{\phi_{G_{j_L}}^{(n)}}$ (resp. $\Sigma_{\phi_{G_{j_R}}^{(n)}}$) denote the singular locus of a germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of corank m , $1 \leq m \leq 3$, on the differentiable function $\phi_{g_{j_L}}^{(n)}(x_{g_{j_\delta}})$ (resp. $\phi_{g_{j_R}}^{(n)}(x_{g_{j_\delta}})$) and let $D_{\Sigma_{\phi_{G_{j_L}}}}$ (resp. $D_{\Sigma_{\phi_{G_{j_R}}}}$) be the neighbourhood of this singular locus.

Then, it can be asserted that:

- 1) the limit set $L(G_L)$ (resp. $L(G_R)$) of the Kleinian group G_L (resp. G_R) corresponds to the singular locus $\Sigma_{\phi_{G_{j_L}}^{(n)}}$ (resp. $\Sigma_{\phi_{G_{j_R}}^{(n)}}$).
- 2) the ordinary set $\Omega(G_L)$ (resp. $\Omega(G_R)$) of G_L (resp. G_R), characterized by a hyperbolic metric, corresponds to the neighbourhood $D_{\Sigma_{\phi_{G_{j_L}}}}$ (resp. $D_{\Sigma_{\phi_{G_{j_R}}}}$) of the singular locus.

Proof.

- 1) The limit set $L(G_L)$ (resp. $L(G_R)$) has a measure equal to zero and, thus, corresponds, by the Sard lemma, to the singular locus $\Sigma_{\phi_{G_{j_L}}^{(n)}}$ (resp. $\Sigma_{\phi_{G_{j_R}}^{(n)}}$).

Furthermore, $L(G_L)$ (resp. $L(G_R)$) is a nowhere dense set: so, we have that:

$$L(G_L) = \Sigma_{\phi_{G_{j_L}}^{(n)}} \quad (\text{resp. } L(G_R) = \Sigma_{\phi_{G_{j_R}}^{(n)}}).$$

- 2) It results from section 2.3.3 that the ordinary set $\Omega(G_L)$ (resp. $\Omega(G_R)$) of G_L (resp. G_R) is characterized by a hyperbolic metric and, thus, that $\Omega(G_L)$ (resp. $\Omega(G_R)$) corresponds to the neighbourhood $D_{\Sigma_{\phi_{g_{j_L}}}}$ (resp. $D_{\Sigma_{\phi_{g_{j_R}}}}$) of the singular locus according to proposition 2.3.1. ■

2.3.5 Corollary

The hyperbolic geometry, characterizing the neighbourhood $D_{\Sigma_{\phi_{g_{j_L}}}}$ (resp. $D_{\Sigma_{\phi_{g_{j_R}}}}$) of the singular locus, in such a way that:

$$\Omega(G_L) = D_{\Sigma_{\phi_{g_{j_L}}}} \quad (\text{resp. } \Omega(G_R) = D_{\Sigma_{\phi_{g_{j_R}}}}),$$

results from the sequence of contracting surjective morphisms of singularization as developed, for example, in proposition 2.1.10.

2.3.6 The unfolded stratum $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$)

Let $\phi_{j_{\delta}}(\omega_L)$ (resp. $\phi_{j_{\delta}}(\omega_R)$) denote a singular germ of corank “ m ” and codimension “ s ” on a n -dimensional differentiable function $\phi_{G_{j_L}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_{\delta}}})$).

Let $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) denote the versal unfolding of the singular germ $\phi_{j_{\delta}}(\omega_L)$ (resp. $\phi_{j_{\delta}}(\omega_R)$).

The dimension of $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) is in general equal to $d_{f_{j_{\delta}}} = s$, where $m \leq s \leq n$.

The unfolded function $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) is embedded in the function $\phi_{G_{j_L}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_{\delta}}})$) such that the complementary $f_{j_{\delta_L}}^{\perp}$ (resp. $f_{j_{\delta_R}}^{\perp}$) of $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) on $\phi_{G_{j_L}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_{\delta}}})$) has dimension $d_{f_{j_{\delta}}^{\perp}} = n - s$ where $s \leq n$.

The neighbourhood of $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) in $\phi_{G_{j_L}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_{\delta}}})$) affected by the versal deformation is denoted $D_{f/\phi_{G_{j_L}}^{(n)}}$ (resp. $D_{f/\phi_{G_{j_R}}^{(n)}}$).

Finally, let $\Sigma_{\phi_{g_{j_L}}}$ (resp. $\Sigma_{\phi_{g_{j_R}}}$) denote the possible singular locus on the quotient algebra of the unfolded function $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) and let $D_{\Sigma_{\phi_{g_{j_L}}}}$ (resp. $D_{\Sigma_{\phi_{g_{j_R}}}}$) be the neighbourhood of this singular locus.

2.3.7 Proposition

The neighbourhood $D_{f/\phi_{G_{j_L}}^{(n)}}$ (resp. $D_{f/\phi_{G_{j_R}}^{(n)}}$) of the unfolded function $f_{j_{\delta_L}}$ (resp. $f_{j_{\delta_R}}$) in $\phi_{G_{j_L}}^{(n)}(x_{g_{j_{\delta}}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_{\delta}}})$) is characterized by a spherical geometry except in the neighbourhood $D_{\Sigma_{f_{j_{\delta_L}}}}$ (resp. $D_{\Sigma_{f_{j_{\delta_R}}}}$) of the singular locus $\Sigma_{\phi_{g_{j_L}}}$ (resp. $\Sigma_{\phi_{g_{j_R}}}$) where the geometry is of hyperbolic type.

Proof.

- 1) The left and right cases will not be distinguished as in proposition 2.3.1.

Let M be a point of coordinates (u^1, \dots, u^n) on the differentiable function $\phi^{(n)}$ in such a way that M is not localized on the singular locus Σ and its neighbourhood D_Σ (see proposition 2.3.1 for the notations).

$(\vec{e}_1, \dots, \vec{e}_n)$ will denote the basis vectors of the proper referential centred on M .

As the stratum $\phi^{(n)} \setminus \Sigma \setminus D_\Sigma$ of $\phi^{(n)}$, being not affected by the singular locus, is Euclidean, the differentials of M and \vec{e}_i are given by:

$$dM = du^i e_i, \quad d\vec{e}_i = \Gamma_{ij}^k du^j \vec{e}_k.$$

The partial derivatives of the coordinates x^i of a point P in the neighbourhood of M will be:

$$D_r x^i = \frac{\partial x^i}{\partial u^r} + \delta_r^i + x^k \Gamma_{kr}^i.$$

- 2) In consequence of the versal deformation of the singular locus Σ , the basis vectors $(\vec{e}_1, \dots, \vec{e}_n)$ will be increased by a small amount:

$$\delta \vec{e}_j = \kappa g_{kj} du^k \cdot M(u^1, \dots, u^n), \quad \text{with } \kappa \in \mathbb{R}^+.$$

In the dimensions of

$$(f_{j\delta} \cup D_{f/\phi^{(n)}}) \setminus (\Sigma \cup D_\Sigma)$$

(as assumed in this proposition),

we then have that:

$$d\vec{e}_j = \Gamma_{ij}^k du^i \vec{e}_k + \kappa g_{jk} du^k \cdot M(u^1, \dots, u^n)$$

leading to:

$$D_r x^j = \frac{\partial x^j}{\partial u^r} + \delta_r^j + x^k \Gamma_{kr}^j + \kappa g_{jk} \delta_r^k.$$

Proceeding as in proposition 2.3.1 to calculate the integrability conditions of $d\vec{e}_j$, we find that the coefficients g_{jk} correspond to a spherical metric of curvature $+\kappa > 0$.

The spherical geometry on $(f_{j\delta} \cup D_{f/\phi^{(n)}}) \setminus (\Sigma \cup D_\Sigma)$ is a consequence of the versal deformation of Σ leading to an over compactness of these strata as resulting from proposition 2.2.8.

- 3) In the neighbourhood $D_{\Sigma_{\phi_{g_{jL}}}}$ of the singular locus $\Sigma_{\phi_{g_{jL}}}$, the geometry is hyperbolic, as developed in proposition 2.3.1. ■

3 Spreading-out isomorphism and strange attractors

3.1 The spreading-out isomorphism

The aim of this chapter is to introduce the blow-up of the versal deformation: it will be called spreading-out (isomorphism) and it is the analogue of the desingularization, also called a monoidal transformation. So, in the prolongation of the singularization and of the versal deformation, the spreading-out isomorphism corresponds to the blow-up of a contracting morphism.

3.1.1 Characteristics of the versal deformation

We refer to propositions 2.2.5 and 2.2.6 where the versal deformation of the semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) of differentiable functions $\phi_{j_\delta}(x_{g_{j_\delta L}})$ (resp. $\phi_{j_\delta}(x_{g_{j_\delta R}})$) endowed with singular germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of corank 1 is given by the contracting fibre bundle:

$$\begin{aligned} D_{S_L} : \quad \theta_{G_L^{(n)}}^* \times \theta_{S_L} &\longrightarrow \theta_{G_L^{(n)}}^* \\ (\text{resp. } D_{S_R} : \quad \theta_{G_R^{(n)}}^* \times \theta_{S_R} &\longrightarrow \theta_{G_R^{(n)}}^*), \end{aligned}$$

in such a way that the fibre

$$\begin{aligned} \theta_{S_L} &= \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_L^i), \dots, \theta^1(\omega_L^s)\} \\ (\text{resp. } \theta_{S_R} &= \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}), \end{aligned}$$

given by the family of sheaves of the base S_L (resp. S_R) of the versal deformation, is projected onto the $(n-1)$ -dimensional coefficient sheaf:

$$\begin{aligned} \theta_L(a) &= \{\theta_L^{n-1}(a_1), \dots, \theta_L^{n-1}(a_i), \dots, \theta_L^{n-1}(a_s)\} \\ (\text{resp. } \theta_R(a) &= \{\theta_R^{n-1}(a_1), \dots, \theta_R^{n-1}(a_i), \dots, \theta_R^{n-1}(a_s)\}) \end{aligned}$$

whose sections $a_{ij_\delta}(x_L) \in \theta_L^{n-1}(a_i)$ (resp. $a_{ij_\delta}(x_R) \in \theta_R^{n-1}(a_i)$), $1 \leq j_\delta \leq r$, are ideals of functions on $\phi_{j_\delta}(x_{g_{j_\delta L}})$ (resp. $\phi_{j_\delta}(x_{g_{j_\delta R}})$).

3.1.2 Lemma

The semisheaves $\theta_L^{n-1}(a_i)$ (resp. $\theta_R^{n-1}(a_i)$), $1 \leq i \leq s$, and $\theta_L^1(\omega_L^i)$ (resp. $\theta_R^1(\omega_R^i)$) are characterized by the same set of ranks.

Proof.

- 1) The section $a_{ij_\delta}(x_L) \subset \phi_{j_\delta}(x_{g_{j_\delta L}})$ being a subfunction of $\phi_{j_\delta}(x_{g_{j_\delta L}})$ must be characterized by a rank

$$n_{a_{ij_\delta}} = (h_{j_\delta} \cdot N)^{n-1}$$

where:

- the integer h_{j_δ} is a global residue degree verifying $h_{j_\delta} > j_\delta$, with j_δ being the global residue degree of the conjugacy class $g_L^{(n)}[j_\delta]$ (see section 1.6) on which $\phi_{j_\delta}(x_{g_{j_\delta L}})$ is defined.
- Note that the rank $r_{g_L^{(n)}}$ of $g_L^{(n)}[j_\delta]$ is $r_{g_L^{(n)}} = (j_\delta \cdot N)^n$, $j_\delta \equiv j \in \mathbb{N}$ [Pie1].
- N is the rank of a real irreducible central completion (see section 1.1).

- 2) As θ_{S_L} (resp. θ_{S_R}) is projected onto $\theta_L(a)$ (resp. $\theta_R(a)$) in such a way that the semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) be flat onto the semisheaf $\theta_L^{n-1}(a_i)$ (resp. $\theta_R^{n-1}(a_i)$), the normal crossing divisor $\omega_{j_\delta L}^i \in \theta_L^1(\omega_L^i)$ (resp. $\omega_{j_\delta R}^i \in \theta_R^1(\omega_R^i)$) must have a rank $n_{\omega_{j_\delta}^i}$ proportional or equal to the rank $n_{a_{ij_\delta}}$ of $a_{ij_\delta}(x_L)$. If $(n-1) \leq 2$, then we have that $n_{\omega_{j_\delta}^i} = (h_{j_\delta} \cdot N)^p$, where $p \geq n-1$.

Remark that we extend here the concept of rank of a (semi)module to the (semi)sheaf defined on this (semi)module.

- 3) Finally, let $n_{\omega^i} = \{n_{\omega_1^i}, \dots, n_{\omega_{j_\delta}^i}, \dots, n_{\omega_r^i}\}$ be the set of ranks of the base semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) and let $n_{a_i} = \{n_{a_{i1}}, \dots, n_{a_{ij_\delta}}, \dots, n_{a_{ir}}\}$ be the corresponding set of ranks of the semisheaf $\theta_L^{n-1}(a_i)$ (resp. $\theta_R^{n-1}(a_i)$) such that, if $n-1 \leq 2$, $n_{\omega_{j_\delta}^i} \geq n_{a_{ij_\delta}}$ in the sense of 1) and 2). ■

3.1.3 Galois antiautomorphic (semi)group

- a) Let $\text{Gal}(\widetilde{F}_L^+/F^0) = \text{Aut}_{F^0} \widetilde{F}_L^+$ (resp. $\text{Gal}(\widetilde{F}_R^+/F^0) = \text{Aut}_{F^0} \widetilde{F}_R^+$) be the Galois automorphic group acting transitively on the set of ideals

$$\widetilde{F}_{v_1}^+ \subset \dots \subset \widetilde{F}_{v_{j_\delta}}^+ \subset \dots \subset \widetilde{F}_{v_r}^+ \quad (\text{resp. } \widetilde{F}_{\overline{v}_1}^+ \subset \dots \subset \widetilde{F}_{\overline{v}_{j_\delta}}^+ \subset \dots \subset \widetilde{F}_{\overline{v}_r}^+)$$

forming an increasing sequence characterized by the extension degrees:

$$[\widetilde{F}_{v_{j_\delta}}^+ : F^0] \equiv [\widetilde{F}_{\overline{v}_{j_\delta}}^+ : F^0] = * + j \cdot N \quad (\text{see section 1.1})$$

and, more particularly, by their global residue degrees forming the increasing sequence:

$$f_{v_{1\delta}} \subset \dots \subset f_{v_{j_\delta}} \subset \dots \subset f_{v_{r\delta}} \quad (\text{resp. } f_{\overline{v}_{1\delta}} \subset \dots \subset f_{\overline{v}_{j_\delta}} \subset \dots \subset f_{\overline{v}_{r\delta}})$$

where: $f_{v_{j\delta}} \equiv f_{\bar{v}_{j\delta}} = j$, $j \in \mathbb{N}$.

- b) Inversely, we introduce the Galois antiautomorphic group $\text{Gal}^{-1}(\tilde{F}_L^+/F^0) = \text{Aut}_{F^0}^{-1} \tilde{F}_L^+$ (resp. $\text{Gal}^{-1}(\tilde{F}_R^+/F^0) = \text{Aut}_{F^0}^{-1} \tilde{F}_R^+$) acting transitively on the set of ideals:

$$\tilde{F}_{v_{s\delta}}^+ \supset \dots \supset \tilde{F}_{v_{j\delta}}^+ \supset \dots \supset \tilde{F}_{v_{1\delta}}^+ \quad (\text{resp. } \tilde{F}_{\bar{v}_{s\delta}}^+ \supset \dots \supset \tilde{F}_{\bar{v}_{j\delta}}^+ \supset \dots \supset \tilde{F}_{\bar{v}_{1\delta}}^+)$$

forming a decreasing sequence characterized by a decreasing chain of global residue degrees:

$$f_{v_{s\delta}} \supset \dots \supset f_{v_{j\delta}} \supset \dots \supset f_{v_{1\delta}} \quad (\text{resp. } f_{\bar{v}_{s\delta}} \supset \dots \supset f_{\bar{v}_{j\delta}} \supset \dots \supset f_{\bar{v}_{1\delta}})$$

with the condition that $s \leq r$.

3.1.4 n -dimensional representations of products, right by left, of Galois groups

- a) Referring to [Pie1], we introduce the explicit n -dimensional representation:

$$\text{Rep}_{\text{Gal}_{\tilde{F}_{R \times L}}}^{(n)} : \text{Gal}(\tilde{F}_L^+/F^0) \times \text{Gal}(\tilde{F}_L^+/F^0) \longrightarrow G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$$

of the product, right by left, of the Galois automorphic groups in such a way that the conjugacy class representatives of the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ form the increasing sequence:

$$g_{R \times L}^{(n)}[1] \subset \dots \subset g_{R \times L}^{(n)}[j\delta] \subset \dots \subset g_{R \times L}^{(n)}[r]$$

characterized by the increasing sequence of their ranks

$$(1 \cdot N)^{2n} \subset \dots \subset (j \cdot N)^{2n} \subset \dots \subset (r \cdot N)^{2n} , \quad j < r \quad (\text{see section 1.6}).$$

- b) Similarly, we can introduce the (inverse) n -dimensional representation:

$$\text{Rep}_{\text{Gal}_{\tilde{F}_{R \times L}}}^{(n)-1} : \text{Gal}^{-1}(\tilde{F}_L^+/F^0) \times \text{Gal}^{-1}(\tilde{F}_L^+/F^0) \longrightarrow G^{-1(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$$

of the product, right by left, of the Galois antiautomorphic groups in such a way that the conjugacy class representatives of the inverse bilinear algebraic semigroup form the decreasing sequence:

$$g_{R \times L}^{(n)}[r] \supset \dots \supset g_{R \times L}^{(n)}[j\delta] \supset \dots \supset g_{R \times L}^{(n)}[s+1]$$

characterized by the decreasing chain of the ranks

$$(r \cdot N)^{2n} \supset \dots \supset (j \cdot N)^{2n} \supset \dots \supset ((s+1) \cdot N)^{2n} ,$$

with the condition that $s \leq r$.

3.1.5 Proposition

Let $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ be the bilinear algebraic semigroup over the product, right by left, of the sets of real completions $F_{\bar{v}}^+$ and F_v^+ .

Then, the inverse bilinear algebraic semigroup $G^{-1(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$ with entries in the product, right by left, of the sets of real completions over the first s places, $s \leq r$, generates the following smooth endomorphism:

$$E[G^{(n)}(F_{\bar{v}}^+ \times F_v^+)] = G^{-1(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+) \oplus G_I^{(n)}(F_{\bar{v}(r-s)}^+ \times F_{v(r-s)}^+)$$

where $G_I^{(n)}(F_{\bar{v}(r-s)}^+ \times F_{v(r-s)}^+)$ is a bilinear algebraic semigroup complementary of $G^{(-1)(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$ in the sense that it is defined with entries in the $(r-s)$ last places.

Proof. As $G^{-1(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$ is the n -dimensional representation of the product, right by left, of Galois antiautomorphic groups according to sections 3.1.3 and 3.1.4, its conjugacy classes form a decreasing sequence from the r -th biplace $\bar{v}_r \times v_r$ until the $(s+1)$ -th biplace $\bar{v}_{s+1} \times v_{s+1}$ in such a way that $(r-s)$ conjugacy classes of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ have been disconnected and generate the complementary bilinear algebraic semigroup $G_I^{(n)}(F_{\bar{v}(r-s)}^+ \times F_{v(r-s)}^+)$.

So, every smooth endomorphism $E[G^{(n)}(F_{\bar{v}}^+ \times F_v^+)]$ of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is generated by the inverse bilinear semigroup $G^{(-1)(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$ in such a way that two non connected bilinear algebraic semigroups $G^{(-1)(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)$ and $G_I^{(n)}(F_{\bar{v}(r-s)}^+ \times F_{v(r-s)}^+)$ are produced from $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$. ■

3.1.6 Corollary

Let $\theta_{G_{R \times L}^{(n)}} = \theta_{G_R^{(n)}} \otimes \theta_{G_R^{(n)}}$ be the bisemisheaf of rings on the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ as introduced in section 1.7.

If $\theta_{G_{R \times L}^{(n)}}$ is noted $\theta_{G^{(n)}(F_{\bar{v}}^+ \times F_v^+)}$, every smooth endomorphism of it is given by:

$$E[\theta_{G^{(n)}(F_{\bar{v}}^+ \times F_v^+)}] = \theta_{G^{-1(n)}(F_{\bar{v}(s)}^+ \times F_{v(s)}^+)} \oplus \theta_{G_I^{(n)}(F_{\bar{v}(r-s)}^+ \times F_{v(r-s)}^+)}.$$

Proof. This is an adaptation of proposition 3.1.5 to the bisemisheaf $\theta_{G^{(n)}(F_{\bar{v}}^+ \times F_v^+)}$ on the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$. ■

3.1.7 Decomposition of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ in irreducible completions

As the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is constructed on products, right by left, of irreducible completions of rank N and as its j_δ -th, m_{j_δ} -th conjugacy class representative $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ counts j_δ products, right by left, of irreducible completions of rank N , we have, in fact, in $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$:

$$r_{\bar{v} \times v}^{nr} = \bigoplus_{j_\delta=1}^r (j_\delta \bullet m^{(j_\delta)})^n$$

pairs of irreducible completions where $m^{(j_\delta)} = \sup m_{j_\delta}$ is the multiplicity of the j_δ -th conjugacy class representative.

So, on $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$, we have an increasing sequence:

$$\begin{aligned} G^{(n)}(F_{\bar{v}_{1_\delta}}^+ \times F_{v_{1_\delta}}^+)_{\text{up}} &\subset \cdots \subset G^{(n)}(F_{\bar{v}_{j_\delta, m_{j_\delta}}}^+ \times F_{v_{j_\delta, m_{j_\delta}}}^+)_{\text{up}} \\ &\subset \cdots \subset G^{(n)}(F_{\bar{v}_{r_\delta, m(r_\delta)}}^+ \times F_{v_{r_\delta, m(r_\delta)}}^+)_{\text{up}} \end{aligned}$$

of sets of conjugacy class representatives where: $G^{(n)}(F_{\bar{v}_{j_\delta, m_{j_\delta}}}^+ \times F_{v_{j_\delta, m_{j_\delta}}}^+)_{\text{up}}$ denotes a bilinear algebraic semigroup whose upper entry is the irreducible bicompletion $(F_{\bar{v}_{j_\delta, m_{j_\delta}}}^+ \times F_{v_{j_\delta, m_{j_\delta}}}^+)$. To simplify the notations, the increasing global residue degree of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$, associated with its structure in products, right by left, of irreducible completions, will be noted f running from 1 to $r_{\bar{v} \times v}^{nr}$.

3.1.8 The fibre θ_{S_L} (resp. θ_{S_R}) of the versal deformation

These considerations can be transposed to the fibre

$$\begin{aligned} \theta_{S_L} &= \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_L^i), \dots, \theta^1(\omega_L^s)\} \\ (\text{resp. } \theta_{S_R} &= \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}) \end{aligned}$$

of the versal deformation in the following way:

Let $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) be the i -th sheaf of the base S_L (resp. S_R) of the versal deformation.

This i -th (semi)sheaf $\theta^1(\omega_L^i)$ (and $\theta^1(\omega_R^i)$) is characterized by the set of ranks $n_{\omega^i} = \{n_{\omega_{1_\delta}^i}, \dots, n_{\omega_{j_\delta}^i}, \dots, n_{\omega_{r_\delta}^i}\}$ according to lemma 3.1.2 where $n_{\omega_{j_\delta}^i}$ refers to the rank of the normal crossings divisor $\omega_{j_\delta}^i$, which is the i -th generator of the versal unfolding of a singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of corank 1 and codimension s on the j_δ -th differentiable function $\phi_{j_\delta}(x_{g_{j_\delta L}})$ (resp. $\phi_{j_\delta}(x_{g_{j_\delta R}})$) of the semisheaf $\theta_{G_L}^*$ (resp. $\theta_{G_R}^*$) (see section 3.1.1 and proposition 2.2.8).

To this rank $n_{\omega_{j_\delta}^i} = (h_{j_\delta} \cdot N)^p$ (see lemma 3.1.2) corresponds the unramified rank or global residue degree $f_{\omega_{j_\delta}^i} = (h_{j_\delta})^p = n_{\omega_{j_\delta}^i} / N^p$ which is the number of irreducible completions on the divisor $\omega_{j_\delta}^i$.

As in section 3.1.7, we shall label the set of irreducible completions in $\theta^1(\omega_L^i)$ (and on $\theta^1(\omega_R^i)$) by a unique integer f_i running over all the normal crossings divisor $\omega_{j_\delta}^i$, $1 \leq j_\delta \leq r$, in such a way that the maximal value of f_i will be given by:

$$f_i^{\max} = \bigoplus_{j_\delta=1}^r \bigoplus_{m_{j_\delta}} (h_{j_\delta, m_{j_\delta}})^p, \quad f_i^{\max} \in \mathbb{N}.$$

So, we have f_i^{\max} irreducible completions of rank N on the i -th base semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) of the versal deformation.

Remark that an integral irreducible closed subscheme of rank N is defined on an irreducible completion of rank N , the concept of rank being extended here from the topological (sub)space to the (sub)scheme on which it is defined.

As a smooth endomorphism was introduced on the bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$ in proposition 3.1.5 and on the bisemisheaf $\theta_{G_{R \times L}^{(n)}}$ on it in corollary 3.1.6, a smooth endomorphism can be defined on the i -th base semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) as follows:

3.1.9 Proposition

Let $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) be the i -th base semisheaf of the versal deformation of the semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$).

Let f_i^{\max} be the maximal value of its global residue degree counting the irreducible closed subschemes of rank N .

Then, the following smooth endomorphism

$$E_{\omega_L^i}[\theta^1(\omega_L^i)_{f_i^{\max}}] = \theta^{*1}(\omega_L^i)_{f_i^*} \oplus \theta_I^1(\omega_L^i)_{f_i^I}, \quad \text{with } f_i^I = f_i^{\max} - f_i^* \in \mathbb{N},$$

can be introduced on the semisheaf $\theta^1(\omega_L^i)_{f_i^{\max}}$ in such a way that it decomposes into two non connected complementary semisheaves $\theta^{*1}(\omega_L^i)_{f_i^*}$ and $\theta^1(\omega_L^i)_{f_i^I}$ whose global residue degrees verify:

$$f_i^{\max} = f_i^* + f_i^I.$$

Proof. The semisheaf $\theta^{*1}(\omega_L^i)_{f_i^*}$ is a “reduced” semisheaf generated from the semisheaf $\theta^1(\omega_L^i)_{f_i^{\max}}$ under the action of the Galois antiautomorphic group according to the endomorphism:

$$E_{\omega_L^i} : \theta^1(\omega_L^i)_{f_i^{\max}} \longrightarrow \theta^{*1}(\omega_L^i)_{f_i^*} \oplus \theta_I^1(\omega_L^i)_{f_i^I}$$

where:

- $\theta_I^i(\omega_L^i)_{f_i^I}$ is the semisheaf complementary of $\theta^{*1}(\omega_L^i)_{f_i^*}$ in the sense of proposition 3.1.5 and corollary 3.1.6.
- $\theta^{*1}(\omega_L^i)_{f_i^*}$ is characterized by decreasing global residue degrees f_i^* while $\theta_I^1(\omega_L^i)_{f_i^I}$ is characterized by increasing global residue degrees f_i^I in such a way that

$$f_i^{\max} = f_i^* + f_i^I, \quad 0 \leq f_i^* \leq f_i^{\max}, \quad 0 \leq f_i^I \leq f_i^{\max}. \quad \blacksquare$$

3.1.10 Proposition

Every base semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) of the versal deformation of the semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$), $1 \leq i \leq s$, can generate under the smooth endomorphism $E_{\omega_L^i}$ (resp. $E_{\omega_R^i}$) the elements of the category $c(\theta_{\omega_L^i}^1)$ (resp. $c(\theta_{\omega_R^i}^1)$) of the $(f_i - 1)$ pairs of semisheaves of rings:

$$c(\theta_{\omega_L^i}^1) = \{(\theta^{*1}(\omega_L^i)_{f_i^{\max}-1} \oplus \theta_I^1(\omega_L^i)_1), \dots, (\theta^{*1}(\omega_L^i)_{f_i^*} \oplus \theta_I^1(\omega_L^i)_{f_i^I}), \\ \dots, (\theta^{*1}(\omega_L^i)_1 \oplus \theta_I^1(\omega_L^i)_{f_i^{\max}-1})\}, \quad 1 \leq f_i^* \leq f_i^{\max},$$

whose objects are two non connected semisheaves characterized by complementary global residue degrees verifying:

$$f_i^{\max} = f_i^* + f_i^I.$$

Proof. This is a generalization of proposition 3.1.9 where $(f_i^{\max} - 1)$ endomorphisms $E_{\omega_L^i}$ are considered. \blacksquare

3.1.11 Corollary

Let f_i^* denote the global residue degree of the reduced semisheaf $\theta^{*1}(\omega_L^i)_{f_i^*}$, $0 \leq f_i^* \leq f_i^{*\max}$. Then, the smooth endomorphism $E_{\omega_L^i}$ is maximal when $f_i^* = 0$.

Proof. If $f_i^* = 0$, then the semisheaf $\theta^1(\omega_L^i)_{f_i^{\max}}$ has been completely transformed under $E_{\omega_L^i}$ into the complementary base semisheaf $\theta_I^1(\omega_L^i)_{f_i^{\max}}$: this is equivalent to say that the base semisheaf $\theta^1(\omega_L^i)_{f_i^{\max}}$ has been totally disconnected from the semisheaf $\theta_{G_L^{(n)}}^*$. \blacksquare

3.1.12 Proposition

Let $\theta_{G_L^{(n)}}^{\text{vers}} = \theta_{G_L^{(n)}}^* \times \theta_{S_L}$ (resp. $\theta_{G_R^{(n)}}^{\text{vers}} = \theta_{G_R^{(n)}}^* \times \theta_{S_R}$) be the versal deformation of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$), as introduced in proposition 2.2.6, where

$$\theta_{S_L} = \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_L^s)\}$$

$$(\text{resp. } \theta_{S_R} = \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}),$$

fibre of the contracting fibre bundle D_{S_L} (resp. D_{S_R}), is the family of the semisheaves of the base S_L (resp. S_R) of the versal deformation.

Then, there exists a family of isomorphisms

$$\Pi_{s_L}(f_1^I, \dots, f_i^I, \dots, f_s^I) :$$

$$\theta_{G_L^{(n)}}^* \times \theta_{S_L} \longrightarrow \theta_{G_L^{(n)}}^* \times \theta'_{S_L} \cup \{\theta_I^1(\omega_L^1)_{f_1^I}, \dots, \theta_I^1(\omega_L^i)_{f_i^I}, \dots, \theta_I^1(\omega_L^s)_{f_s^I}\}$$

$$(\text{resp. } \Pi_{s_R}(f_1^I, \dots, f_i^I, \dots, f_s^I) :$$

$$\theta_{G_R^{(n)}}^* \times \theta_{S_R} \longrightarrow \theta_{G_R^{(n)}}^* \times \theta'_{S_R} \cup \{\theta_I^1(\omega_R^1)_{f_1^I}, \dots, \theta_I^1(\omega_R^i)_{f_i^I}, \dots, \theta_I^1(\omega_R^s)_{f_s^I}\})$$

disconnecting f_1^I irreducible subsheaves of rank N from the base semisheaf $\theta^1(\omega_L^1)_{f_1^{\max}}$ on $\theta_{G_L^{(n)}}^*$, ..., f_i^I irreducible subsheaves of rank N from the base semisheaf $\theta^1(\omega_L^i)_{f_i^{\max}}$ on $\theta_{G_L^{(n)}}^*$, ..., and so on, $1 \leq i \leq s$.

The set of complementary global residue degrees $(f_1^I, \dots, f_i^I, \dots, f_s^I)$ varies in such a way that $1 \leq f_1^I \leq f_1^{\max}, \dots, 1 \leq f_i^I \leq f_i^{\max}, \dots, 1 \leq f_s^I \leq f_s^{\max}$ implying, for each set $(f_1^I, \dots, f_i^I, \dots, f_s^I)$ a family of isomorphisms $\Pi_s(f_1^I, \dots, f_i^I, \dots, f_s^I)$.

The residue fibre θ'_{S_L} (resp. θ'_{S_R}) is given by:

$$\theta'_{S_L} = \theta_{S_L} \setminus \{\theta_I^1(\omega_L^1)_{f_1^I}, \dots, \theta_I^1(\omega_L^i)_{f_i^I}, \dots, \theta_I^1(\omega_L^s)_{f_s^I}\}$$

$$(\text{resp. } \theta'_{S_R} = \theta_{S_R} \setminus \{\theta_I^1(\omega_R^1)_{f_1^I}, \dots, \theta_I^1(\omega_R^i)_{f_i^I}, \dots, \theta_I^1(\omega_R^s)_{f_s^I}\}).$$

Proof. This proposition is a generalization of proposition 3.1.10 in such a way that the smooth endomorphism $E_{\omega_L^i}$ (resp. $E_{\omega_R^i}$), generating $(f_i^{\max} - 1)$ pairs of semisheaves of the category $c(\theta_{\omega_L^i}^1)$ (resp. $c(\theta_{\omega_R^i}^1)$), is extended to all the base semisheaves $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$), $1 \leq i \leq s$, of the considered versal deformation. ■

3.1.13 Corollary

The family of isomorphisms $\Pi_{s_L}(f_1^I, \dots, f_i^I, \dots, f_s^I)$ (resp. $\Pi_{s_R}(f_1^I, \dots, f_i^I, \dots, f_s^I)$) is maximal in the i -th semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$), and, then, noted $\Pi_{s_L}^{\max(i)}(f_1^I, \dots, f_i^I, \dots, f_s^I)$ (resp. $\Pi_{s_R}^{\max(i)}(f_1^I, \dots, f_i^I, \dots, f_s^I)$), if $f_i^* = 0$.

Proof. If $f_i^* = 0$, then the base semisheaf $\theta^1(\omega_L^i)$ has been completely transformed, under $E_{\omega_L^i}$, into the disconnected complementary semisheaf $\theta_I^1(\omega_L^i)_{f_i^I}$ in such a way that $f_i^I = f_i^{\max}$. ■

3.1.14 Corollary

The family isomorphisms $\Pi_{s_L}^{\max}(f_1^I, \dots, f_i^I, \dots, f_s^I)$ (resp. $\Pi_{s_R}^{\max}(f_1^I, \dots, f_i^I, \dots, f_s^I)$) is maximal if it is maximal in every semisheaf $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$), $1 \leq i \leq s$, of the base of the versal deformation.

Proof. This is a generalization of corollary 3.1.3 to all the base semisheaves $\theta^1(\omega_L^i)$ (resp. $\theta^1(\omega_R^i)$) of the versal deformation, implying that:

- 1) • $f_1^* = 0$ and $f_1^I = f_1^{\max}$
 \vdots
 • $f_i^* = 0$ and $f_i^I = f_i^{\max}$
 \vdots
 • $f_s^* = 0$ and $f_s^I = f_s^{\max}$

$$2) \quad \Pi_{s_L}^{\max}(f_1^I, \dots, f_i^I, \dots, f_s^I) :$$

$$\theta_{G_L^{(n)}}^* \times \theta_{S_L} \rightarrow \theta_{G_L^{(n)}}^* \cup \{\theta_I^1(\omega_L^1)_{f_1^{\max}}, \dots, \theta_I^1(\omega_L^i)_{f_i^{\max}}, \dots, \theta_I^1(\omega_L^s)_{f_s^{\max}}\}$$

$$(\text{resp.} \quad \Pi_{s_R}^{\max}(f_1^I, \dots, f_i^I, \dots, f_s^I) :$$

$$\theta_{G_R^{(n)}}^* \times \theta_{S_R} \rightarrow \theta_{G_R^{(n)}}^* \cup \{\theta_I^1(\omega_R^1)_{f_1^{\max}}, \dots, \theta_I^1(\omega_R^i)_{f_i^{\max}}, \dots, \theta_I^1(\omega_R^s)_{f_s^{\max}}\}). \blacksquare$$

3.1.15 Category of vertical tangent bundles

Let $T_{V_{W_L}} = \{T_{V_{W_L^1}}, \dots, T_{V_{W_L^i}}, \dots, T_{V_{W_L^s}}\}$ (resp. $T_{V_{W_R}} = \{T_{V_{W_R^1}}, \dots, T_{V_{W_R^i}}, \dots, T_{V_{W_R^s}}\}$) denote the family of tangent vector bundles obtained by the projection of all the disconnected base semisheaves $\theta_I^1(\omega_L^i)$ (resp. $\theta_I^1(\omega_R^i)$), $1 \leq i \leq s$, in the vertical tangent spaces $T_{V_{W_L^i}}$ (resp. $T_{V_{W_R^i}}$) characterized by normal vector fields \vec{W}_{i_L} (resp. \vec{W}_{i_R}).

The proper projective map of the tangent bundle $T_{V_{W_L^i}}$ (resp. $T_{V_{W_R^i}}$) is given by:

$$\begin{aligned} \tau_{V_{W_L^i}} : \quad T_{V_{W_L^i}}(\theta_I^1(\omega_L^i)_{f_i^I}) &\longrightarrow \theta_I^1(\omega_L^i)_{f_i^I} \\ (\text{resp. } \tau_{V_{W_R^i}} : \quad T_{V_{W_R^i}}(\theta_I^1(\omega_R^i)_{f_i^I}) &\longrightarrow \theta_I^1(\omega_R^i)_{f_i^I}) \end{aligned}$$

so that $\tau_{V_{W_L}} = \{\tau_{V_{W_L^i}}\}_{i=1}^s$ (resp. $\tau_{V_{W_R}} = \{\tau_{V_{W_R^i}}\}_{i=1}^s$).

To the category $c(\theta_I^1(\omega_L^i))$ (resp. $c(\theta_I^1(\omega_R^i))$) of disconnected base semisheaves $\theta_I^1(\omega_L^i)$ (resp. $\theta_I^1(\omega_R^i)$), $1 \leq i \leq s$, will then correspond the category $c(T_{V_{W_L^i}}(\theta_I^1(\omega_L^i)))$ (resp. $c(T_{V_{W_R^i}}(\theta_I^1(\omega_R^i)))$) of sections of tangent vector bundles.

3.1.16 Proposition

The extension of the quotient algebra $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) of the versal deformation of the singular semisheaf $\theta_{G_L^{(n)}}^$ (resp. $\theta_{G_R^{(n)}}^*$), having an isolated degenerate singularity of corank 1 and codimension s in each section of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$), is realized by the spreading-out isomorphism*

$$SOT_L = (\tau_{V_{W_L}} \circ \Pi_{s_L}) \quad (\text{resp. } SOT_R = (\tau_{V_{W_R}} \circ \Pi_{s_R})).$$

Proof. Let $I_{\omega_L^i}$ (resp. $I_{\omega_R^i}$) be the kernel of the normal vector bundle $T_{V_{W_L^i}}$ (resp. $T_{V_{W_R^i}}$). Then, the exact sequence:

$$\begin{aligned} 0 \longrightarrow I_{\omega_L^i} \longrightarrow T_{V_{W_L^i}}(\theta_I^1(\omega_L^i)_{f_i^I}) &\xrightarrow{\tau_{V_{W_L^i}}} \theta_I^1(\omega_L^i)_{f_i^I} \longrightarrow 0 \\ (\text{resp. } 0 \longrightarrow I_{\omega_R^i} \longrightarrow T_{V_{W_R^i}}(\theta_I^1(\omega_R^i)_{f_i^I}) &\xrightarrow{\tau_{V_{W_R^i}}} \theta_I^1(\omega_R^i)_{f_i^I} \longrightarrow 0) \end{aligned}$$

represents an extension of $\theta_I^1(\omega_L^i)_{f_i^I}$ (resp. $\theta_I^1(\omega_R^i)_{f_i^I}$) by the kernel $I_{\omega_L^i}$ (resp. $I_{\omega_R^i}$).

And, the isomorphism $SOT_L = (\tau_{V_{W_L}} \circ \Pi_{s_L})$ (resp. $SOT_R = (\tau_{V_{W_R}} \circ \Pi_{s_R})$) constitutes an extension of the quotient algebra $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) of the versal deformation of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) (see definition 2.2.7) since the base semisheaf θ_{S_L} (resp. θ_{S_R}) has been partially or completely extracted from $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$): this also corresponds to an extension of the desingularization process as it will be described in the next chapter. ■

3.1.17 Corollary

The extension of the quotient algebra of the versal deformation of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) is maximal if the spreading-out isomorphism is given by:

$$SOT_L^{\max} = (\tau_{V_{W_L}} \circ \Pi_{s_L}^{\max}) \quad (\text{resp. } SOT_R^{\max} = (\tau_{V_{W_R}} \circ \Pi_{s_R}^{\max})).$$

Proof. Indeed, in this case, the base semisheaf θ_{S_L} (resp. θ_{S_R}) has been completely pulled out from the quotient algebra $\theta[\omega_L]$ (resp. $\theta[\omega_R]$) in the sense of corollary 3.1.14 and projected in the vertical tangent space $T_{V_{W_L}}$ (resp. $T_{V_{W_R}}$) according to the map $\tau_{V_{W_L}}$ (resp. $\tau_{V_{W_R}}$). ■

3.1.18 Proposition

The spreading-out isomorphism SOT_L (resp. SOT_R) is locally stable if the generated disconnected semisheaves $T_{V_{W_L}^i}(\theta_I^1(\omega_L^i)_{f_i^I})$ (resp. $T_{V_{W_R}^i}(\theta_I^1(\omega_R^i)_{f_i^I})$) are locally free semisheaves.

Proof. That is to say that the semisheaves $T_{V_{W_L}^i}(\theta_I^1(\omega_L^i)_{f_i^I})$ (resp. $T_{V_{W_R}^i}(\theta_I^1(\omega_R^i)_{f_i^I})$) are free of singularities. ■

3.1.19 Proposition

The maximal number of complementary disconnected semisheaves $T_{V_{W_L}^i}(\theta_I^1(\omega_L^i)_{f_i^I})$ (resp. $T_{V_{W_R}^i}(\theta_I^1(\omega_R^i)_{f_i^I})$) is equal to s .

Proof. Indeed, the integer “ s ” is the codimension of the versal deformation of the semisheaves $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$). ■

3.1.20 Gluing-up disconnected base semisheaves

Let $\theta_i^1(D\omega_L^i)$ (resp. $\theta_i^1(D\omega_R^i)$) and $\theta_j^1(D\omega_L^j)$ (resp. $\theta_j^1(D\omega_R^j)$) denote the i -th and j -th complementary semisheaves $T_{V_{W_L}^i}(\theta_I^1(\omega_L^i)_{f_i^I})$ (resp. $T_{V_{W_R}^i}(\theta_I^1(\omega_R^i)_{f_i^I})$) and $T_{V_{W_L}^j}(\theta_I^1(\omega_L^j)_{f_j^I})$ (resp. $T_{V_{W_R}^j}(\theta_I^1(\omega_R^j)_{f_j^I})$) extracted from the base semisheaf θ_{S_L} (resp. θ_{S_R}). These semisheaves can be glued together in a compact way according to:

For each pair (i, j) , let Π_{ij} be an isomorphism from $\theta_j^1(D(\omega_L^i) \cap D(\omega_L^j))$ to $\theta_i^1(D(\omega_L^i) \cap D(\omega_L^j))$ where $D(\omega_L^i)$ and $D(\omega_L^j)$ denote the domains on which these semisheaves $\theta_i^1(D\omega_L^i)$ and $\theta_j^1(D\omega_L^j)$ are respectively defined.

Then, there exists a semisheaf $\theta^1(D(\omega_L^{i-j}))$, defined on the connected domain $D(\omega_L^{i-j}) = D(\omega_L^i) \cup D(\omega_L^j)$, and an isomorphism n_i from $\theta^1(D\omega_L^i)$ to $\theta_i^1(D\omega_L^i)$ such that $\Pi_{ij} = n_i \circ n_j^{-1}$ in each point of $D(\omega_L^i) \cap D(\omega_L^j)$, $\forall i, j$, $1 \leq i, j \leq s$: this is an adapted version of a proposition of J.P. Serre [Ser1].

So, $\theta^1(D(\omega_L^{i-j}))$ is the semisheaf corresponding to the gluing-up of the semisheaves $\theta_i^1(D\omega_L^i)$ and $\theta_j^1(D\omega_L^j)$.

Note that the right case “ R ” can be handled similarly and parallelly.

3.1.21 Sequence of spreading-out isomorphisms

Let $\theta_{SOT(1)_L}$ (resp. $\theta_{SOT(1)_R}$) denote the family of disconnected base semisheaves $\{\theta_i^1(D\omega_L^i)\}_{i=1}^s$ (resp. $\{\theta_i^1(D\omega_R^i)\}_{i=1}^s$) of the extension of the quotient algebra of the versal deformation $SOT(1)_L$ (resp. $SOT(1)_R$). This family of semisheaves, having been glued together according to section 3.1.20, covers partially the product $\theta_{G_L^{(n)}}^* \times \theta'_{S_L}$ (resp. $\theta_{G_R^{(n)}}^* \times \theta'_{S_R}$) of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) by the residue fibre θ'_{S_L} (resp. θ'_{S_R}) of the versal deformation having not been disconnected from $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) by the isomorphisms $\Pi_{S_L}(f_1^I, \dots, f_i^I, \dots, f_s^I) \in SOT(1)_L$ (resp. $\Pi_{S_R}(f_1^I, \dots, f_i^I, \dots, f_s^I) \in SOT(1)_R$).

If the spreading-out isomorphism $SOT(1)_L$ (resp. $SOT(1)_R$) is not locally stable, as noted in proposition 3.1.18, then singular germs $\omega_{j_{\delta_L}}^i$ (resp. $\omega_{j_{\delta_R}}^i$), $1 \leq i \leq s$, $1 \leq j_{\delta} \leq r$, on the sections of the base semisheaves $\theta_{SOT(1)_L}$ (resp. $\theta_{SOT(1)_R}$) can be degenerated.

Consequently, a versal deformation of $\theta_{SOT(1)_L}$ (resp. $\theta_{SOT(1)_R}$) can be envisaged followed by a spreading-out isomorphism $SOT(2)_L$ (resp. $SOT(2)_R$). The resulting family of disconnected base semisheaves of the extension of the quotient algebra of the versal deformation $SOT(2)_L$ (resp. $SOT(2)_R$) will be noted $\theta_{SOT(2)_L}$ (resp. $\theta_{SOT(2)_R}$).

So, a set of “ h ” versal deformations followed by “ h ” spreading-out isomorphisms can be envisaged until the disconnected base semisheaves $\theta_{SOT(h)_L}$ (resp. $\theta_{SOT(h)_R}$) are free or locally stable, $h \leq (s-1)$, $h \in \mathbb{N}$.

3.1.22 Remark

The spreading-out isomorphism was envisaged for singular semisheaves $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) having isolated singularities of corank 1. If it is referred to section 2.2.3 where the versal deformation of germs of corank 2 is considered, it is not difficult to develop the spreading-out isomorphism for unfolded germs of corank 2 (and corank 3) similarly as it was done for germs of corank 1.

3.2 Inner dynamics and strange attractors

This section envisages the spreading-out isomorphism from a differentiable dynamical point of view in such a way that strange attractors, related to the versal deformation of singular germs, blow up under the spreading-out isomorphism into new disconnected attractors.

3.2.1 Left and right diffeomorphisms of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$

The generation of the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ can also be studied from a diffeomorphic point of view leading to an inner bidynamics. In this respect, the differentiable baction of the product, right by left, $F_{\bar{v}}^+ \times F_v^+$ of the sets of completions of the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is considered. This baction is a homomorphism:

$$F_{\bar{v}}^+ \times F_v^+ \longrightarrow \text{Diff}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$$

in such a way that

$$(F_{\bar{v}}^+ \times F_v^+) \times G^{(n)}(F_{\bar{v}}^+ \times F_v^+) \longrightarrow G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$$

is differentiable.

$\text{Diff}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$ denotes the group of all diffeomorphisms of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ splitting into:

$$\text{Diff}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+)) = \text{Diff}_R(G_R^{(n)}(F_{\bar{v}}^+)) \times \text{Diff}_L(G_L^{(n)}(F_v^+))$$

where $\text{Diff}_L(G^{(n)}(F_v^+))$ (resp. $\text{Diff}_R(G^{(n)}(F_{\bar{v}}^+))$) denotes the semigroup of left (resp. right) diffeomorphisms.

If these diffeomorphisms are studied from the point of view of orbit structure, then a left (resp. right) generator $f_L \in \text{Diff}_L(G_L^{(n)}(F_v^+))$ (resp. $f_R \in \text{Diff}_R(G_R^{(n)}(F_{\bar{v}}^+))$) must be taken into account as acting on an irreducible completion $F_{v_{1_\delta}^+}$ (resp. $F_{\bar{v}_{1_\delta}^+}$) of rank N in such a way that the orbits of $F_{v_{1_\delta}^+}$ (resp. $F_{\bar{v}_{1_\delta}^+}$) relative to f_L (resp. f_R) are the left (resp. right) subsets $\{f_L^{j_\delta}(F_{v_{1_\delta}^+})\}_{j_\delta=1}^r$ (resp. $\{f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+})\}_{j_\delta=1}^r$) of the j_δ conjugacy classes of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_{\bar{v}}^+)$).

3.2.2 Proposition

A set of $(j_\delta)^n$ left (resp. right) orbits $f_L^{j_\delta}(F_{v_{1_\delta}^+})$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+})$) of $F_{v_1^+}$ (resp. $F_{\bar{v}_1^+}$) relative to f_L (resp. f_R) constitutes the structure of the (j_δ, m_{j_δ}) -th conjugacy class representative $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_{\bar{v}}^+)$).

Each orbit $f_L^{j_\delta}(F_{v_{1_\delta}^+}^+)$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+}^+)$) is composed of j_δ irreducible completions $F_{v_{j_\delta}^{j'_\delta}}^+$ (resp. $F_{\bar{v}_{j_\delta}^{j'_\delta}}^+$), $1 \leq j'_\delta \leq j_\delta$, of rank N and is associated with a Frobenius substitution:

$$f_L(F_{v_{1_\delta}^+}^+) \longrightarrow f_L^{j_\delta}(F_{v_{1_\delta}^+}^+) \quad (\text{resp.} \quad f_R(F_{\bar{v}_{1_\delta}^+}^+) \longrightarrow f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+}^+)).$$

Proof.

a) As the left (resp. right) orbit $f_L^{j_\delta}(F_{v_{1_\delta}^+}^+)$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+}^+)$) is the image of the map:

$$f_L^{j_\delta} : F_{v_{1_\delta}^+}^+ \longrightarrow g_L^{(n)}[j_\delta, m_{j_\delta}] \quad (\text{resp.} \quad f_R^{j_\delta} : F_{\bar{v}_{1_\delta}^+}^+ \longrightarrow g_R^{(n)}[j_\delta, m_{j_\delta}]),$$

it must correspond to a divisor at j_δ irreducible completions $F_{v_{j_\delta}^{j'_\delta}}^+$ (resp. $F_{\bar{v}_{j_\delta}^{j'_\delta}}^+$), $1 \leq j'_\delta \leq j_\delta$, of rank N according to [Pie1].

As a result, the Frobenius substitution:

$$f_L \longrightarrow f_L^{j_\delta} \quad (\text{resp.} \quad f_R \longrightarrow f_R^{j_\delta})$$

on the generator $f_L \in \text{Diff}_L(G_L^{(n)}(F_v^+))$ (resp. $f_R \in \text{Diff}_R(G_R^{(n)}(F_{\bar{v}}^+))$) follows, expressing that we are dealing with one-dimensional components of representatives of the j_δ -th conjugacy class of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_{\bar{v}}^+)$).

b) As the rank of the conjugacy class representative $g_L^{(n)}[j_\delta, m_{j_\delta}]$ is $r_{v_{j_\delta}}^{(n)} = (j_\delta \cdot N)^n$ and its global residue is $f_{v_{j_\delta}}^{(n)} = j_\delta^n$, the number of left (resp. right) orbits $f_L^{j_\delta}(F_{v_{1_\delta}^+}^+)$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}^+}^+)$) of $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) must be equal to

$$n_{O_{f_L^{j_\delta}}} \equiv n_{O_{f_R^{j_\delta}}} = (j_\delta)^n. \quad \blacksquare$$

3.2.3 Proposition

Let $\text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$ denote the group of outer automorphisms of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$. If the bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is assumed to be (C^r) -differentiable, then the isomorphism:

$$I_{\text{Out} \rightarrow \text{Diff}} : \text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+)) \longrightarrow \text{Diff}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$$

follows.

Proof. According to [Pie1], the group of outer automorphisms $\text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$ of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ is given by:

$$\text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+)) = \text{Aut}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+)) / \text{Int}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$$

with $\text{Int}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+)) = \text{Aut}(P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+))$ where $\text{Aut}(P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+))$ is the group of automorphisms of the bilinear parabolic semigroup $P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+)$.

As a result, $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ acts on $P^{(n)}(F_{\bar{v}^1}^+ \times F_{v^1}^+)$ by conjugation generating by this way the conjugacy class representatives $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$.

And, these conjugacy class representatives $g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ can be decomposed into normal crossing completions resulting from their compactification as developed in [Pie1].

So, we have that:

$$g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}] \simeq \prod_n (F_{\bar{v}_{j_\delta}, m_{j_\delta}}^+ \times F_{v_{j_\delta}, m_{j_\delta}}^+)$$

implying that $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) can be partitioned into $(j_\delta)^n$ completions from an algebraic point of view. Indeed, it has been seen in [Pie1] that

$$\text{Out}(g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]) \simeq \prod_n (\text{Gal}(\tilde{F}_{\bar{v}_{j_\delta}, m_{j_\delta}}^+ / F^0) \times \text{Gal}(\tilde{F}_{v_{j_\delta}, m_{j_\delta}}^+ / F^0))$$

where $\text{Out}(g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]) \subset \text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$.

Thus, a one-to-one correspondence can be established between the elements of $\text{Out}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$ and the elements of $\text{Diff}(G^{(n)}(F_{\bar{v}}^+ \times F_v^+))$ leading to the generation of the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) which decompose into completions from an algebraic point of view or into orbits from a differentiable point of view in such a way that they correspond bijectively. ■

3.2.4 Inner bidynamics

- 1) By a one parameter semigroup of left (resp. right) diffeomorphisms $\text{Diff}_L(G_L^{(n)}(F_v^+))$ (resp. $\text{Diff}_R(G_L^{(n)}(F_v^+))$) of the algebraic semigroup $G_L^{(n)}(F_v^+)$ (resp. $G_L^{(n)}(F_{\bar{v}}^+)$), we mean a continuous map:

$$f_L^{j_\delta} : F_{v_{j_\delta}^1}^+ \times g_L^{(n)}[j_\delta, m_{j_\delta}] \longrightarrow g_L^{(n)}[j_\delta, m_{j_\delta}]$$

$$(\text{resp. } f_R^{j_\delta} : g_R^{(n)}[j_\delta, m_{j_\delta}] \times F_{\bar{v}_{j_\delta}^1}^+ \longrightarrow g_R^{(n)}[j_\delta, m_{j_\delta}])$$

for every conjugacy class representative of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) such that, for all $k_\delta, \ell_\delta \in \mathbb{N}$ verifying $j_\delta = k_\delta + \ell_\delta$, we have that:

$$f_{L;k_\delta+\ell_\delta}^{j_\delta}(x_{j_{\delta_L}^{(1)}}) = f_{L;k_\delta}^{j_\delta} \cdot f_{L;\ell_\delta}^{j_\delta}(x_{j_{\delta_L}^{(1)}}), \quad x_{j_{\delta_L}^{(1)}} \in F_{v_{j_\delta}^+}^+$$

$$(\text{resp. } f_{R;k_\delta+\ell_\delta}^{j_\delta}(x_{j_{\delta_R}^{(1)}}) = f_{R;k_\delta}^{j_\delta} \cdot f_{R;\ell_\delta}^{j_\delta}(x_{j_{\delta_R}^{(1)}}), \quad x_{j_{\delta_R}^{(1)}} \in F_{\bar{v}_{j_\delta}^+}^+),$$

where k_δ and ℓ_δ refer to the numbers of irreducible completions in $f_L^{j_\delta}$ (resp. $f_R^{j_\delta}$).

- 2) The tangent (semi)space to the one-parameter semigroup of left (resp. right) diffeomorphisms of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) is the space of sections $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) of the tangent bundle $T(G_L^{(n)}(F_v^+))$ (resp. $T(G_R^{(n)}(F_v^+))$) whose fibres in each point $x_{j_{\delta_L}} \in g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $x_{j_{\delta_R}} \in g_R^{(n)}[j_\delta, m_{j_\delta}]$) are given by the tangent vectors $\vec{n}_L(x_{j_{\delta_L}})$ (resp. $\vec{n}_R(x_{j_{\delta_R}})$) defined by:

$$\vec{n}_L(x_{j_{\delta_L}}) = \frac{d}{dt} \left(f_{L;t}^{j_\delta}(x_{j_{\delta_L}^{(1)}}) \right)_{t=0}$$

$$(\text{resp. } \vec{n}_R(x_{j_{\delta_R}}) = \frac{d}{dt} \left(f_{R;t}^{j_\delta}(x_{j_{\delta_R}^{(1)}}) \right)_{t=0}).$$

This allows to generate a left (resp. right) internal dynamics of the algebraic semigroup $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$) taking into account that the tangent bundle $T(G_L^{(n)}(F_v^+))$ (resp. $T(G_R^{(n)}(F_v^+))$) has to be horizontal in order that the tangent vectors $\vec{n}_L(x_{j_{\delta_L}})$ (resp. $\vec{n}_R(x_{j_{\delta_R}})$) be rotational velocity vectors.

- 3) An internal bidynamics of the bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$ can be reached by considering the horizontal tangent bibundle

$$T(G^{(n)}(F_v^+ \times F_v^+)) = T(G_R^{(n)}(F_v^+)) \times T(G_L^{(n)}(F_v^+))$$

whose (bi)fibres in each bipoint $x_{j_{\delta_R}} \times x_{j_{\delta_L}} \in g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]$ are the tangent bivectors $\vec{n}_R(x_{j_{\delta_R}}) \times \vec{n}_L(x_{j_{\delta_L}})$.

3.2.5 Translated orbits in the neighbourhood of singular germs

We are now interested by the dynamics around singularities on the sections $\phi_{G_{j_{\delta_L}}}^{\text{TAN}}$ (resp. $\phi_{G_{j_{\delta_R}}}^{\text{TAN}}$) of the space of sections $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) of the tangent bundle $T(G_L^{(n)}(F_v^+))$ (resp. $T(G_R^{(n)}(F_v^+))$) on the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$).

Let then $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) denote a singular germ of corank 1 and codimension s on the n -dimensional real-valued differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}}) \in \Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}}) \in T(G_R^{(n)}(F_{\bar{v}}^+))$) on $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$).

As $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) can be decomposed into a set of left (resp. right) orbits $f_L^{j_\delta}(F_{v_{1_\delta}}^+)$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}}^+)$) relative to $f_L \in \text{Diff}(G_L^{(n)}(F_v^+))$ (resp. $f_R \in \text{Diff}(G_R^{(n)}(F_{\bar{v}}^+))$), the differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}$) can also be decomposed into one-dimensional subfunctions $f_{L;\text{TAN}}^{j_\delta}$ (resp. $f_{R;\text{TAN}}^{j_\delta}$) corresponding to orbits translated from $f_L^{j_\delta}(F_{v_{1_\delta}}^+)$ (resp. $f_R^{j_\delta}(F_{\bar{v}_{1_\delta}}^+)$) under the tangent bundle $T(G_L^{(n)}(F_v^+))$ (resp. $T(G_R^{(n)}(F_{\bar{v}}^+))$):

$$T(G_L^{(n)}(F_v^+)) : f_{L;\text{TAN}}^{j_\delta} \longrightarrow f_L^{j_\delta}, \quad (\text{resp. } T(G_R^{(n)}(F_{\bar{v}}^+)) : f_{R;\text{TAN}}^{j_\delta} \longrightarrow f_R^{j_\delta}).$$

As we are concerned with a singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) on the differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$) which is perturbed in the neighbourhood $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) of $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$), it shall be assumed that the “translated” orbits on $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) are perturbed one-dimensional subfunctions $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) of $f_{L;\text{TAN}}^{j_\delta}$ (resp. $f_{R;\text{TAN}}^{j_\delta}$) characterized by a rank $r_{f_{L;\text{TAN}}^{a_{j_\delta}}} = r_{f_{R;\text{TAN}}^{a_{j_\delta}}} = a \cdot N$ where the global residue degree a is inferior or equal to j_δ : $a \leq j_\delta$.

Before going closely into the study of the neighbourhoods of singular germs from a diffeomorphic point of view, the tangent space decomposition into contracting and expanding components will be recalled.

3.2.6 Splitting of the tangent space into stable, unstable and neutral subsets

Let $f_{L;\text{TAN}} \in \text{Diff}_L(T(G_L^{(n)}(F_v^+)))$ (resp. $f_{R;\text{TAN}} \in \text{Diff}_R(T(G_R^{(n)}(F_{\bar{v}}^+)))$) denote the left (resp. right) generator of the diffeomorphisms of the space of sections of the tangent bundle $T(G_L^{(n)}(F_v^+))$ (resp. $T(G_R^{(n)}(F_{\bar{v}}^+))$).

- A point $x_{g_{j_\delta L}}^{\text{TAN}; N\omega} \in \phi_{G_{j_\delta L}}^{\text{TAN}}$ (resp. $x_{g_{j_\delta R}}^{\text{TAN}; N\omega} \in \phi_{G_{j_\delta R}}^{\text{TAN}}$) is non-wandering if, for every neighbourhood $U_{\phi_{j_\delta L}^{\text{TAN}}}$ (resp. $U_{\phi_{j_\delta R}^{\text{TAN}}}$) of $x_{g_{j_\delta L}}^{\text{TAN}; N\omega}$ (resp. $x_{g_{j_\delta R}}^{\text{TAN}; N\omega}$), one has:

$$f_{L;\text{TAN}}^{j_\delta}(U_{\phi_{j_\delta L}^{\text{TAN}}}) \cap U_{\phi_{j_\delta L}^{\text{TAN}}} \neq \emptyset$$

$$(\text{resp. } f_{R;\text{TAN}}^{j_\delta}(U_{\phi_{j_\delta R}^{\text{TAN}}}) \cap U_{\phi_{j_\delta R}^{\text{TAN}}} \neq \emptyset).$$

The set of non wandering points forms a closed invariant set noted $\Omega_{\phi_{j_\delta L}^{\text{TAN}}}$ (resp. $\Omega_{\phi_{j_\delta R}^{\text{TAN}}}$). The other points of $\phi_{G_{j_\delta L}}^{\text{TAN}}$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}$) are called wandering points and form invariant open subsets [Sma].

- A linear automorphism u_L (resp. u_R) of the tangent space $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) will be said to be flat, contracting or expanding if, under

$$u_L : \Gamma(T(G_L^{(n)}(F_v^+))) \longrightarrow \Gamma(T(G_L^{(n)}(F_v^+)))$$

$$(\text{resp. } u_R : \Gamma(T(G_R^{(n)}(F_v^+))) \longrightarrow \Gamma(T(G_R^{(n)}(F_v^+))) ,$$

the eigenvalues of

$$|u_L(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))| = \lambda^{j_\delta} |f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}})| , \quad \forall 1 \leq j_\delta \leq r ,$$

$$(\text{resp. } |u_R(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))| = \lambda^{j_\delta} |f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}})| , \quad \forall 1 \leq j_\delta \leq r ,)$$

satisfy respectively $|\lambda^{j_\delta}| = 1$, $|\lambda^{j_\delta}| < 1$ or $|\lambda^{j_\delta}| > 1$.

So, the tangent space $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) exhibits a splitting into:

- stable subsets $E^s(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^s(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) for which u_L (resp. u_R) is contracting.
- unstable subsets $E^u(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^u(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) for which u_L (resp. u_R) is expanding.
- neutral subsets $E^n(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^n(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) for which u_L (resp. u_R) is flat.

Classically, a subset being stable and unstable, i.e. exhibiting respectively a volume contraction and a volume expansion under a linear automorphism of the tangent space, is said to be “hyperbolic”. This terminology will not be adopted here as it will appear in the following.

3.2.7 Proposition

Let $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) denote the space of sections of the tangent bundle to the algebraic semigroup $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_v^+)$). Then we have that:

- 1) its stable subsets $E^s(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^s(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) are characterized by a hyperbolic geometry.
- 2) its unstable subsets $E^u(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^u(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) are characterized by a spherical geometry.

3) its neutral subsets $E^n(f_{L;\text{TAN}}^{j\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))$ (resp. $E^n(f_{R;\text{TAN}}^{j\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))$) are characterized by an euclidian geometry.

Proof. With reference to proposition 2.3.1, the developments will be envisaged for the left and right cases without distinction and the notations will be simplified as follows:

Let the point $x_{j_{\delta_L}^{(1)}}^{\text{TAN}}$ (and $x_{j_{\delta_R}^{(1)}}^{\text{TAN}}$) be given by a point M of coordinates (u^1, \dots, u^n) and let $f_{L;\text{TAN}}^{j\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}})$ (and $f_{R;\text{TAN}}^{j\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}})$) define a point P of coordinates (x^1, \dots, x^n) in the neighbourhood of M .

The differential $d(\overrightarrow{MP})$ of the vector \overrightarrow{MP} corresponds to a linear automorphism:

$$\begin{aligned} u : \quad \Gamma(T(G_{L,R}^{(n)}(F_v^+))) &\longrightarrow \Gamma(T(G_{L,R}^{(n)}(F_v^+))) \\ \overrightarrow{MP} &\longrightarrow d(\overrightarrow{MP}) \end{aligned}$$

where $G_{L,R}^{(n)}(F_v^+)$ is a condensed notation for $G_L^{(n)}(F_v^+)$ or $G_R^{(n)}(F_v^+)$.

$d(\overrightarrow{MP})$ can be expressed by the differential [Car]

$$dP = (Dx^1, \dots, Dx^i, \dots, Dx^n)$$

where $Dx^i = dx^i + du^i + x^k \Gamma_{kr}^i du^r$, $1 \leq i \leq n$, with

$$\Gamma_{kr}^i = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial u^r} + \frac{\partial g_{ir}}{\partial u^k} + \frac{\partial g_{kr}}{\partial u^i} \right).$$

Let then h be the covering of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ by the euclidian space \mathbb{R}^n :

$$h : \quad \Gamma(T(G_{L,R}^{(n)}(F_v^+))) \longrightarrow \mathbb{R}^n$$

given by:

$$dP \longrightarrow h(dP)$$

in such a way that:

$$\begin{aligned} h \circ u : \quad \Gamma(T(G_{L,R}^{(n)}(F_v^+))) &\longrightarrow \mathbb{R}^n, \\ \overrightarrow{MP} &\longrightarrow h(d(\overrightarrow{MP})). \end{aligned}$$

Three possibilities occur:

1) if $\|\overrightarrow{MP}\| = \|h(d(\overrightarrow{MP}))\|$, the subsets of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ are locally Euclidian.

It follows that the norm of \overrightarrow{MP} is conserved under the composition of maps $(h \circ u)$ and, thus, that the linear automorphism u of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ is unitary.

Consequently, in:

$$|u_L(f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}}))| = \lambda^{j_\delta} |f_{L;\text{TAN}}^{j_\delta}(x_{j_{\delta_L}^{(1)}}^{\text{TAN}})|, \quad (*)$$

$$(\text{resp. } |u_R(f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}}))| = \lambda^{j_\delta} |f_{R;\text{TAN}}^{j_\delta}(x_{j_{\delta_R}^{(1)}}^{\text{TAN}})|),$$

- a) $|\lambda^{j_\delta}| = 1$.
- b) u_L (resp. u_R) is a flat automorphism characterized by an euclidian geometry on the subsets of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ which are neutral.
- 2) if $\|\overrightarrow{MP}\| > \|h(d(\overrightarrow{MP}))\|$, one must admit, according to proposition 2.3.1, that $d(\overrightarrow{MP})$ is given by the differential of P whose components Dx^i are:

$$Dx^i = dx^i + du^i + x^k \Gamma_{kr}^i du^r - \kappa g_{ik} du^k \quad \text{with } \kappa \in \mathbb{R}.$$

The metric g_{ik} is locally hyperbolic and the curvature of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ is locally negative or equal to $-\kappa$.

The norm \overrightarrow{MP} is not conserved under $(h \circ u)$.

Thus, in (*), we have that:

- a) $|\lambda^{j_\delta}| < 1$.
- b) u_L (resp. u_R) is a hyperbolic automorphism which is contracting.
- c) the subsets of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ are stable and characterized by a hyperbolic geometry.
- 3) if $\|\overrightarrow{MP}\| < \|h(d(\overrightarrow{MP}))\|$, the components Dx^i of dP are given by:

$$Dx^i = dx^i + du^i + x^k \Gamma_{kr}^i du^r + \kappa g_{ik} du^k.$$

The metric g_{ik} is locally spherical and the curvature of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ is locally positive or equal to $+\kappa$.

Then, in (*), we have that:

- a) $|\lambda^{j_\delta}| > 1$.
- b) u_L (resp. u_R) is a “spherical” automorphism which is expanding.
- c) the subsets of $\Gamma(T(G_{L,R}^{(n)}(F_v^+)))$ are locally unstable and characterized by a spherical geometry. ■

3.2.8 Proposition: Singular hyperbolic attractor

Let $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) be a singular germ (of corank 1 and codimension s) on the n -dimensional real-valued differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}}) \in \Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}}) \in \Gamma(T(G_R^{(n)}(F_v^+)))$).

The neighbourhood $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) of the singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) on $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$) is a singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) with respect to $\text{Diff}_L(T(G_L^{(n)}(F_v^+)))$ (resp. $\text{Diff}_R(T(G_R^{(n)}(F_v^+)))$) provided that:

- 1) the orbits of Λ_L^{TAN} (resp. Λ_R^{TAN}) are one-dimensional functions $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) having a rank $r_{f_{L;\text{TAN}}}^{a_{j_\delta}} = r_{f_{R;\text{TAN}}}^{a_{j_\delta}} = a \cdot N$ and form the basin of attraction of Λ_L^{TAN} (resp. Λ_R^{TAN}).
- 2) the singularity is a non-wandering point.
- 3) the basin of attraction of Λ_L^{TAN} (resp. Λ_R^{TAN}) is a stable subset $E^s(f_{L;\text{TAN}}^{a_{j_\delta}})$ (resp. $E^s(f_{R;\text{TAN}}^{a_{j_\delta}})$) of $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$) characterized by a hyperbolic geometry.

Proof.

- 1) According to section 3.2.5, the orbits on the neighbourhood $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) of the singularity $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) are one-dimensional subfunctions $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) of $f_{L;\text{TAN}}^{j_\delta}$ (resp. $f_{R;\text{TAN}}^{j_\delta}$). Consequently, they constitute the basin of attraction of the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}).
- 2) Let $U_{\phi_{j_\delta L}^{\text{TAN}}} \subset \Lambda_L^{\text{TAN}}$ (resp. $U_{\phi_{j_\delta R}^{\text{TAN}}} \subset \Lambda_R^{\text{TAN}}$) be a neighbourhood of the singularity included into the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}).

Then, the singularity is a non wandering point if we have, according to section 3.2.6:

$$f_{L;\text{TAN}}^{a_{j_\delta}}(U_{\phi_{j_\delta L}^{\text{TAN}}}) \cap U_{\phi_{j_\delta L}^{\text{TAN}}} \neq \emptyset$$

$$(\text{resp. } f_{R;\text{TAN}}^{a_{j_\delta}}(U_{\phi_{j_\delta R}^{\text{TAN}}}) \cap U_{\phi_{j_\delta R}^{\text{TAN}}} \neq \emptyset).$$

- 3) If was proved in proposition 2.3.1 that the geometry is hyperbolic in the neighbourhood of the singularity. Consequently, the basin of attraction of the singular hyperbolic attractor is a stable subset $E^s(f_{L;\text{TAN}}^{a_{j_\delta}})$ (resp. $E^s(f_{R;\text{TAN}}^{a_{j_\delta}})$) characterized by a hyperbolic geometry whose points are contracting under the linear automorphism u_L (resp. u_R) of $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$). ■

3.2.9 The introduction of unfolded attractors

- Let the singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of corank 1 and codimension s be given by $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$).
- Its versal deformation corresponds to the development:

$$F(\omega_L, a_{ij}(x_L)) = \omega_L^{s+2} + \sum_{i=1}^s a_{ij_\delta}(x_L) \omega_{j_\delta L}^i$$

$$(\text{resp. } F(\omega_R, a_{ij}(x_R)) = \omega_R^{s+2} + \sum_{i=1}^s a_{ij_\delta}(x_R) \omega_{j_\delta R}^i)$$

where:

- $a_{ij_\delta}(x_L)$ (resp. $a_{ij_\delta}(x_R)$) is a $(n-1)$ -dimensional real valued differentiable function defined on the neighbourhood $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) of the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) on the n -dimensional differentiable function $\phi_{G_{j_\delta L}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$ (resp. $\phi_{G_{j_\delta R}}^{\text{TAN}}(x_{g_{j_\delta}}^{\text{TAN}})$).
- $\omega_{j_\delta L}^i$ (resp. $\omega_{j_\delta R}^i$) is a divisor, generator of the versal unfolding of $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$). It is localized on $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) and projected on $a_{ij_\delta}(x_L)$ (resp. $a_{ij_\delta}(x_R)$) according to proposition 2.2.8.
- The set $\bigcup_{i=1}^s a_{ij_\delta}(x_L)$ (resp. $\bigcup_{i=1}^s a_{ij_\delta}(x_R)$) of functions on $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$) can be partitioned into the one-dimensional functions $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$), which are perturbed orbits of $\text{Diff}_L(T(G_L^{(n)}(F_v^+)))$ (resp. $\text{Diff}_R(T(G_R^{(n)}(F_v^+)))$) (see proposition 3.2.8).
- As the set $\bigcup_{i=1}^s \omega_{j_\delta L}^i$ (resp. $\bigcup_{i=1}^s \omega_{j_\delta R}^i$) of generators of the versal unfolding of the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) is localized on $D_{\phi_{G_{j_\delta L}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_\delta R}}^{\text{TAN}}}$), it constitutes the basin of an unfolded attractor $\Lambda_{unfL}^{\text{TAN}}$ (resp. $\Lambda_{unfR}^{\text{TAN}}$):
 - a) centred on the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$);
 - b) having as orbits the generators $\omega_{j_\delta L}^i$ (resp. $\omega_{j_\delta R}^i$), $1 \leq i \leq s$, which can be rewritten according to:

$$\omega_{j_\delta L}^i = f_{L;\text{TAN}}^{\omega_{j_\delta}^i}(\omega_{j_\delta L}^{(1)}) \quad (\text{resp. } \omega_{j_\delta R}^i = f_{R;\text{TAN}}^{\omega_{j_\delta}^i}(\omega_{j_\delta R}^{(1)}))$$

where:

- $f_{L;\text{TAN}}^{\omega_{j_\delta}^i}$ (resp. $f_{R;\text{TAN}}^{\omega_{j_\delta}^i}$) is an orbital generator with respect to $\text{Diff}_L(\omega_{j_\delta L}^i)$ (resp. $\text{Diff}_R(\omega_{j_\delta R}^i)$);
- $\omega_{j_\delta L}^{(1)}$ (resp. $\omega_{j_\delta R}^{(1)}$) is a point of an irreducible completion of $\omega_{j_\delta L}^i$ (resp. $\omega_{j_\delta R}^i$).

3.2.10 Proposition

Let Λ_L^{TAN} (resp. Λ_R^{TAN}) be a singular hyperbolic attractor centred on a singular germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) of corank 1 and codimension s .

Then, the versal unfolding of the germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) involves the map:

$$VD_{\Lambda_L} : \Lambda_L^{\text{TAN}} \longrightarrow \Lambda_{str_L}^{\text{TAN}} \quad (\text{resp. } VD_{\Lambda_R} : \Lambda_R^{\text{TAN}} \longrightarrow \Lambda_{str_R}^{\text{TAN}})$$

of the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) into the singular strange attractor

$$\Lambda_{str_L}^{\text{TAN}} = \Lambda_L^{\text{TAN}} \times \Lambda_{unf_L}^{\text{TAN}} \quad (\text{resp. } \Lambda_{str_R}^{\text{TAN}} = \Lambda_R^{\text{TAN}} \times \Lambda_{unf_R}^{\text{TAN}})$$

where $\Lambda_{unf_L}^{\text{TAN}}$ (resp. $\Lambda_{unf_R}^{\text{TAN}}$) is the unfolded attractor introduced in section 3.2.9.

Proof. As the versal unfolding of the germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$), given by $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$), leads to the projection of s divisors $\omega_{j_\delta L}^i$ (resp. $\omega_{j_\delta R}^i$) on the neighbourhood $D_{\phi_{j_\delta L}^{\text{TAN}}}$ (resp. $D_{\phi_{j_\delta R}^{\text{TAN}}}$) of the singularity and as these divisors $\omega_{j_\delta L}^i$ (resp. $\omega_{j_\delta R}^i$) can be rewritten according to $f_{L;\text{TAN}}^{\omega_{j_\delta L}^i}(\omega_{j_\delta(1)})$ (resp. $f_{R;\text{TAN}}^{\omega_{j_\delta R}^i}(\omega_{j_\delta(1)})$) (see section 3.2.9), it is clear that the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) will undergo an unfolding leading to the attractor

$$\Lambda_{str_L}^{\text{TAN}} = \Lambda_L^{\text{TAN}} \times \Lambda_{unf_L}^{\text{TAN}} \quad (\text{resp. } \Lambda_{str_R}^{\text{TAN}} = \Lambda_R^{\text{TAN}} \times \Lambda_{unf_R}^{\text{TAN}})$$

which is a strange attractor.

Indeed, according to proposition 2.3.7, the strata of the neighbourhood of the singularity perturbed by the versal deformation are characterized by a spherical geometry. These strata are the generators (or the orbits) $f_{L;\text{TAN}}^{\omega_{j_\delta L}^i}(\omega_{j_\delta(1)})$ (resp. $f_{R;\text{TAN}}^{\omega_{j_\delta R}^i}(\omega_{j_\delta(1)})$) of the versal deformation projected on the orbits $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) of the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}).

Consequently, these strata are given by the functions:

$$f_{L;\text{TAN}}^{str} = f_{L;\text{TAN}}^{a_{j_\delta}|\omega_{j_\delta L}^i} \times f_{L;\text{TAN}}^{\omega_{j_\delta L}^i} \quad (\text{resp. } f_{R;\text{TAN}}^{str} = f_{R;\text{TAN}}^{a_{j_\delta}|\omega_{j_\delta R}^i} \times f_{R;\text{TAN}}^{\omega_{j_\delta R}^i})$$

where $f_{L;\text{TAN}}^{a_{j_\delta}|\omega_{j_\delta L}^i}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}|\omega_{j_\delta R}^i}$) is a one-dimensional subfunction of $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) characterized by a rank $r_{f_{L;\text{TAN}}^{a_{j_\delta}|\omega_{j_\delta L}^i}} = b \cdot N$ inferior to the rank $r_{f_{L;\text{TAN}}^{a_{j_\delta}}} = a \cdot N$ of $f_{L;\text{TAN}}^{a_{j_\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j_\delta}}$) with $b < a$, $b \in \mathbb{N}$ (see proposition 3.2.8).

These strata $f_{L;\text{TAN}}^{str}$ (resp. $f_{R;\text{TAN}}^{str}$) belong to the basin of the singular strange attractor $\Lambda_{str_L}^{\text{TAN}}$ (resp. $\Lambda_{str_R}^{\text{TAN}}$) in such a way that:

- a) $f_{L;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i} \in \Lambda_L^{\text{TAN}}$ (resp. $f_{R;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i} \in \Lambda_R^{\text{TAN}}$);
 b) $f_{L;\text{TAN}}^{\omega_{j\delta}^i} \in \Lambda_{unf_L}^{\text{TAN}}$ (resp. $f_{R;\text{TAN}}^{\omega_{j\delta}^i} \in \Lambda_{unf_R}^{\text{TAN}}$).

These strata $f_{L;\text{TAN}}^{\text{str}}$ (resp. $f_{R;\text{TAN}}^{\text{str}}$) are characterized by a spherical geometry since their points are expanding under the automorphism u_L (resp. u_R). Consequently, these strata $f_{L;\text{TAN}}^{\text{str}}$ (resp. $f_{R;\text{TAN}}^{\text{str}}$) constitute unstable subsets $E^u(f_{L;\text{TAN}}^{\text{str}})$ (resp. $E^u(f_{R;\text{TAN}}^{\text{str}})$) of $\Gamma(T(G_L^{(n)}(F_v^+)))$ (resp. $\Gamma(T(G_R^{(n)}(F_v^+)))$). ■

3.2.11 Proposition

The simplest singular strange attractor $\Lambda_{str_L}^{\text{TAN}}$ (resp. $\Lambda_{str_R}^{\text{TAN}}$) is composed of:

- 1) $a(n)$ (unfolded) singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$);
- 2) a stable subset $E^s(f_{L;\text{TAN}}^{(a-b)_{j\delta}})$ (resp. $E^s(f_{R;\text{TAN}}^{(a-b)_{j\delta}})$) characterized by a hyperbolic geometry;
- 3) unstable subsets $E^u(f_{L;\text{TAN}}^{\text{str}})$ (resp. $E^u(f_{R;\text{TAN}}^{\text{str}})$) characterized by a spherical geometry.

Proof.

- 1) Let the unstable subsets $E^u(f_{L;\text{TAN}}^{\text{str}})$ (resp. $E^u(f_{R;\text{TAN}}^{\text{str}})$) be given by the functions $f_{L;\text{TAN}}^{\text{str}}$ (resp. $f_{R;\text{TAN}}^{\text{str}}$) according to proposition 3.2.10.

Then, the stable subset $E^s(f_{L;\text{TAN}}^{(a-b)_{j\delta}})$ (resp. $E^s(f_{R;\text{TAN}}^{(a-b)_{j\delta}})$) characterized by a hyperbolic geometry, is:

$$E^s(f_{L;\text{TAN}}^{(a-b)_{j\delta}}) = E^s(f_{L;\text{TAN}}^{a_{j\delta}}) - \bigcup_{i=1}^s f_{L;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i}$$

$$\text{(resp. } E^s(f_{R;\text{TAN}}^{(a-b)_{j\delta}}) = E^s(f_{R;\text{TAN}}^{a_{j\delta}}) - \bigcup_{i=1}^s f_{R;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i} \text{)}$$

where:

- $E^s(f_{L;\text{TAN}}^{a_{j\delta}})$ (resp. $E^s(f_{R;\text{TAN}}^{a_{j\delta}})$) is the basin of attraction of the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) given by the orbits $f_{L;\text{TAN}}^{a_{j\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j\delta}}$) having a rank $r_{f_{L;\text{TAN}}^{a_{j\delta}}} = a \cdot N$ according to proposition 3.2.8.
- $f_{L;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i}$ (resp. $f_{R;\text{TAN}}^{a_{j\delta}|\omega_{j\delta}^i}$) is a subfunction of $f_{L;\text{TAN}}^{a_{j\delta}}$ (resp. $f_{R;\text{TAN}}^{a_{j\delta}}$) on which the generator $\omega_{j\delta_L}^i$ (resp. $\omega_{j\delta_R}^i$) has been projected according to proposition 3.2.10.

- 2) The concept of singular attractor was introduced in [Pie3] and the notion of singular strange attractor was developed in [Pie3] and independently in [P-R].

The fact that $\Lambda_{str_L}^{TAN}$ (resp. $\Lambda_{str_R}^{TAN}$) is a strange attractor corresponds to its description by H. Schuster [Sch]:

“A strange attractor arises typically when the flow contracts the volume element in some directions, but stretches it along others. To remain confined to a bounded domain, the volume element is folded at the same time”.

An excellent literature on strange attractors can also be found in [Rue1], [Rue2]; [Mil2], [M-P], [E-R] and [Wil]. ■

3.2.12 Structure of a general singular strange attractor

According to proposition 3.2.10, a singular strange attractor is defined by:

$$\Lambda_{str_L}^{TAN} = \Lambda_L^{TAN} \times \Lambda_{unf_L}^{TAN} \quad (\text{resp. } \Lambda_{str_R}^{TAN} = \Lambda_R^{TAN} \times \Lambda_{unf_R}^{TAN})$$

where the unfolded attractor $\Lambda_{unf_L}^{TAN}$ (resp. $\Lambda_{unf_R}^{TAN}$) has as orbits the generators $\omega_{j\delta_L}^i$ (resp. $\omega_{j\delta_R}^i$), $1 \leq i \leq s$, of the versal deformation of the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) localized on the singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}).

But, these generators $\omega_{j\delta_L}^i$ (resp. $\omega_{j\delta_R}^i$) can carry singularities for $i \geq 2$ according to proposition 2.2.8 and section 3.1.21. As a result, a generator $\omega_{j\delta_L}^i$ (resp. $\omega_{j\delta_R}^i$) carrying a singularity is a singular hyperbolic attractor, noted $\Lambda_{\omega_{j\delta_L}^i}^{TAN}$ (resp. $\Lambda_{\omega_{j\delta_R}^i}^{TAN}$), according to proposition 3.2.8.

Consequently, the unfolded attractor can be rewritten as follows:

$$\Lambda_{unf_L}^{TAN} = \bigcup_{i=1}^s \Lambda_{\omega_{j\delta_L}^i}^{TAN} \quad (\text{resp. } \Lambda_{unf_R}^{TAN} = \bigcup_{i=1}^s \Lambda_{\omega_{j\delta_R}^i}^{TAN})$$

where:

- $\Lambda_{\omega_{j\delta_L}^i}^{TAN}$ (resp. $\Lambda_{\omega_{j\delta_R}^i}^{TAN}$) is a singular hyperbolic attractor for $i \geq 2$.
- $\Lambda_{\omega_{j\delta_L}^1}^{TAN}$ (resp. $\Lambda_{\omega_{j\delta_R}^1}^{TAN}$) is a divisor.

And, a general singular strange attractor can be decomposed according to:

$$\Lambda_{str_L}^{TAN} = \Lambda_L^{TAN} \times \left(\bigcup_{i=1}^s \Lambda_{\omega_{j\delta_L}^i}^{TAN} \right) \quad (\text{resp. } \Lambda_{str_R}^{TAN} = \Lambda_R^{TAN} \times \left(\bigcup_{i=1}^s \Lambda_{\omega_{j\delta_R}^i}^{TAN} \right)).$$

3.2.13 Proposition

Let $SOT_{j_{\delta_L}}^{\max} = (\tau_{V_{\omega_{j_{\delta_L}}}} \circ \pi_{s_{j_{\delta_L}}}^{\max})$ (resp. $SOT_{j_{\delta_R}}^{\max} = (\tau_{V_{\omega_{j_{\delta_R}}}} \circ \pi_{s_{j_{\delta_R}}}^{\max})$) denote the maximal spreading-out isomorphism pulling out completely the generators $\omega_{j_{\delta_L}}^i$ (resp. $\omega_{j_{\delta_R}}^i$) of the versal deformation of the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) from its neighbourhood $D_{\phi_{G_{j_{\delta_L}}}^{\text{TAN}}}$ (resp. $D_{\phi_{G_{j_{\delta_R}}}^{\text{TAN}}}$).

Then, the maximal spreading-out isomorphism

$$SOT_{j_{\delta_L}}^{\max} : \Lambda_{str_L}^{\text{TAN}} \longrightarrow \Lambda_L^{\text{TAN}} \oplus \bigcup_{i=1}^s \Lambda_{\omega_{j_{\delta_L}}^i}^{\text{TAN}}$$

$$(\text{resp. } SOT_{j_{\delta_R}}^{\max} : \Lambda_{str_R}^{\text{TAN}} \longrightarrow \Lambda_R^{\text{TAN}} \oplus \bigcup_{i=1}^s \Lambda_{\omega_{j_{\delta_R}}^i}^{\text{TAN}})$$

decomposes the general singular strange attractor $\Lambda_{str_L}^{\text{TAN}}$ (resp. $\Lambda_{str_R}^{\text{TAN}}$) into the original singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) and into a set of $(s-1)$ disconnected singular hyperbolic attractors and a disconnected divisor.

Proof. The $SOT_{j_{\delta_L}}^{\max}$ (resp. $SOT_{j_{\delta_R}}^{\max}$) spreading-out isomorphism corresponds to an extension of the quotient algebra of the versal deformation of the singular semisheaf $\theta_{G_L}^{*(n)}$ (resp. $\theta_{G_R}^{*(n)}$) developed in corollary 3.1.17 and restricted to the $(j_{\delta}, m_{j_{\delta}})$ conjugacy class representative of $G_L^{(n)}(F_v^+)$ (resp. $G_R^{(n)}(F_{\bar{v}}^+)$).

As the unfolded attractor $\Lambda_{unf_L}^{\text{TAN}}$ (resp. $\Lambda_{unf_R}^{\text{TAN}}$) is the union of hyperbolic attractors $\Lambda_{\omega_{j_{\delta_L}}^i}^{\text{TAN}}$ (resp. $\Lambda_{\omega_{j_{\delta_R}}^i}^{\text{TAN}}$) and of a divisor, which are the generators $\omega_{j_{\delta_L}}^i$ (resp. $\omega_{j_{\delta_R}}^i$) of the versal deformation of the singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$), the maximal spreading-out isomorphism $SOT_{j_{\delta_L}}^{\max}$ (resp. $SOT_{j_{\delta_R}}^{\max}$) is the inverse map

$$SOT_{j_{\delta_L}}^{\max} = (VD_{\Lambda_L})^{-1} \quad (\text{resp. } SOT_{j_{\delta_R}}^{\max} = (VD_{\Lambda_R})^{-1})$$

of the versal unfolding

$$VD_{\Lambda_L} : \Lambda_L^{\text{TAN}} \longrightarrow \Lambda_{str_L}^{\text{TAN}} \quad (\text{resp. } VD_{\Lambda_R} : \Lambda_R^{\text{TAN}} \longrightarrow \Lambda_{str_R}^{\text{TAN}})$$

introduced in proposition 3.2.10.

Consequently, $SOT_{j_{\delta_L}}^{\max}$ (resp. $SOT_{j_{\delta_R}}^{\max}$) blows up the general singular strange attractor $\Lambda_{str_L}^{\text{TAN}}$ (resp. $\Lambda_{str_R}^{\text{TAN}}$) into the original singular hyperbolic attractor Λ_L^{TAN} (resp. Λ_R^{TAN}), $(s-1)$ disconnected singular hyperbolic attractors $\Lambda_{\omega_{j_{\delta_L}}^i}^{\text{TAN}}$ (resp. $\Lambda_{\omega_{j_{\delta_R}}^i}^{\text{TAN}}$), $i \geq 2$, and a divisor $\Lambda_{\omega_{j_{\delta_L}}^1}^{\text{TAN}}$ (resp. $\Lambda_{\omega_{j_{\delta_R}}^1}^{\text{TAN}}$) corresponding to the generators of the versal deformation. ■

3.2.14 Corollary

Let VD_{Λ_L} (resp. VD_{Λ_R}) denote the versal deformation transforming a hyperbolic singular attractor Λ_L^{TAN} (resp. Λ_R^{TAN}) into a general singular strange attractor $\Lambda_{str_L}^{\text{TAN}}$ (resp. $\Lambda_{str_R}^{\text{TAN}}$).

Let $SOT_{j_{\delta_L}}^{\text{max}}$ (resp. $SOT_{j_{\delta_R}}^{\text{max}}$) be the maximal spreading-out isomorphism blowing up the general singular strange attractor.

Then, the following composition of maps:

$$\begin{aligned} SOT_{j_{\delta_L}}^{\text{max}} \circ VD_{\Lambda_L} : \quad \Lambda_L^{\text{TAN}} &\xrightarrow{VD_{\Lambda_L}} \Lambda_{str_L}^{\text{TAN}} \xrightarrow{SOT_{j_{\delta_L}}^{\text{max}}} \Lambda_L^{\text{TAN}} \oplus \bigcup_{i=1}^s \Lambda_{\omega_{j_{\delta_L}}^i}^{\text{TAN}} \\ (\text{resp. } SOT_{j_{\delta_R}}^{\text{max}} \circ VD_{\Lambda_R} : \quad \Lambda_R^{\text{TAN}} &\xrightarrow{VD_{\Lambda_R}} \Lambda_{str_R}^{\text{TAN}} \xrightarrow{SOT_{j_{\delta_R}}^{\text{max}}} \Lambda_R^{\text{TAN}} \oplus \bigcup_{i=1}^s \Lambda_{\omega_{j_{\delta_R}}^i}^{\text{TAN}}) \end{aligned}$$

is such that

$$SOT_{j_{\delta_L}}^{\text{max}} = (VD_{\Lambda_L})^{-1} \quad (\text{resp. } SOT_{j_{\delta_R}}^{\text{max}} = (VD_{\Lambda_R})^{-1}).$$

Proof. This is a consequence of proposition 3.2.13 and of the generation of the versal deformation given in proposition 2.2.8.

Note that the paper of [B-L-M-P] introduces the explosion of singular cycles, phenomenon closed to the blowing up of strange attractors considered here and in [Pie3].

■

4 Langlands global correspondences affected by degenerate singularities

The aim of this chapter consists in showing in what extent it is possible to develop global correspondences of Langlands for a (bisemi)sheaf of rings on the real bilinear algebraic semigroup affected by degenerate singularities in the sense of chapters 2 and 3.

4.1 The transformation of the bisemisheaf of rings $\theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}}$ on the real bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$ under degenerate singularities

First, it will be recalled what is the n -dimensional real irreducible global correspondence of Langlands as developed in [Pie1]: it consists in a bijection between the n -dimensional irreducible representation $\text{Irr Rep}_{W_{F_R^+ \times F_L^+}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ of the product, right by left, of Weil groups and the irreducible cuspidal representation of $G^{(n)}(F_v^+ \times F_v^+)$ given by $\text{Irr ELLIP}(\text{GL}_n(\mathbb{A}_{F_v^+, T} \times \mathbb{A}_{F_v^+, T}))$. These concepts will thus be reviewed, and, among others, the definition of global Weil groups.

4.1.1 Global Weil groups

Let $\text{Gal}(\tilde{F}_{v_{j_\delta}}^+ / F^0)$ (resp. $\text{Gal}(\tilde{F}_{\bar{v}_{j_\delta}}^+ / F^0)$) denote the Galois subgroup of the extension $\tilde{F}_{v_{j_\delta}}^+$ (resp. $\tilde{F}_{\bar{v}_{j_\delta}}^+$) in one-to-one correspondence with the pseudo-ramified completion $F_{v_{j_\delta}}^+$ (resp. $F_{\bar{v}_{j_\delta}}^+$) having a rank given by:

$$[F_{v_{j_\delta}}^+ : F^0] = * + j_\delta \cdot N, \quad * \in \mathbb{N}, \quad * < N$$

$$(\text{resp. } [F_{\bar{v}_{j_\delta}}^+ : F^0] = * + j_\delta \cdot N).$$

And, let $\text{Gal}(\dot{\tilde{F}}_{v_{j_\delta}}^+ / F^0)$ (resp. $\text{Gal}(\dot{\tilde{F}}_{\bar{v}_{j_\delta}}^+ / F^0)$) be the Galois subgroup of the pseudo-ramified extension $\dot{\tilde{F}}_{v_{j_\delta}}^+$ (resp. $\dot{\tilde{F}}_{\bar{v}_{j_\delta}}^+$) characterized by a degree:

$$[\dot{\tilde{F}}_{v_{j_\delta}}^+ : F^0] = j_\delta \cdot N \quad (\text{resp. } [\dot{\tilde{F}}_{\bar{v}_{j_\delta}}^+ : F^0] = j_\delta \cdot N)$$

(see section 1.1).

Then, according to [Pie1], the global Weil group $W_{F_L^+}^{ab}$ (resp. $W_{F_R^+}^{ab}$) is given by:

$$W_{F_L^+}^{ab} = \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} \text{Gal}(\dot{\tilde{F}}_{v_{j_\delta}, m_{j_\delta}}^+ / F^0) \quad (\text{resp. } W_{F_R^+}^{ab} = \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} \text{Gal}(\dot{\tilde{F}}_{\bar{v}_{j_\delta}, m_{j_\delta}}^+ / F^0))$$

and, its product, right by left, is:

$$W_{F_R^+}^{ab} \times W_{F_L^+}^{ab} = \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} \text{Gal}(\dot{F}_{\bar{v}_{j_\delta}, m_{j_\delta}}^+ / F^0) \times \text{Gal}(\dot{F}_{v_{j_\delta}, m_{j_\delta}}^+ / F^0) .$$

The n -dimensional irreducible representation of the product, right by left, $W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}$ of global Weil groups is given by:

$$\text{Irr Rep}_{W_{F_R^+ \times F_L^+}}^{(n)} (W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) = G^{(n)}(F_{\bar{v}_\oplus}^+ \times F_{v_\oplus}^+)$$

according to [Pie1] (proposition 3.4.3).

4.1.2 Cuspidal representation of $\text{GL}_n(\mathbf{F}_{\bar{v}}^+ \times \mathbf{F}_v^+)$

Let $G^{(n)}(F_{\bar{v}}^{+,T} \times F_v^{+,T})$ be the reductive bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$ submitted to the toroidal compactification $\gamma_{R \times L}^c$ as developed in section 3.4 of [Pie1].

Its conjugacy class representatives $g_{T_{R \times L}}^{(n)}[j_\delta, m_{j_\delta}] = g_{T_R}^{(n)}[j_\delta, m_{j_\delta}] \times g_{T_L}^{(n)}[j_\delta, m_{j_\delta}]$ are products, right by left, of n -dimensional real semitori $T_R^n[j_\delta, m_{j_\delta}] \times T_L^n[j_\delta, m_{j_\delta}]$.

Each complex-valued bifunction $\phi_{G_R^T}^{(n)}(x_{g_{j_\delta R}^T}) \otimes \phi_{G_L^T}^{(n)}(x_{g_{j_\delta L}^T})$ on the conjugacy class representative $g_{T_{R \times L}}^{(n)}[j_\delta, m_{j_\delta}] \in G^{(n)}(F_{\bar{v}}^{+,T} \times F_v^{+,T})$ is given by:

$$\begin{aligned} \phi_{G_R^T}^{(n)}(x_{g_{j_\delta R}^T}) \otimes \phi_{G_L^T}^{(n)}(x_{g_{j_\delta L}^T}) &= T_R^n[j_\delta, m_{j_\delta}] \times T_L^n[j_\delta, m_{j_\delta}] \\ &= \lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{-2\pi i j_\delta x} \otimes \lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{2\pi i j_\delta x} \end{aligned}$$

where:

- $\lambda(n, j_\delta, m_{j_\delta}) = \prod_{c=1}^n \lambda_c(n, j_\delta, m_{j_\delta})$ is a product of eigenvalues $\lambda_c(n, j_\delta, m_{j_\delta})$ of the j_δ -th coset representative $(U_{j_\delta R} \times U_{j_\delta L})$ of the product $(T_R(n; t) \otimes T_L(n; t))$ of Hecke operators;
- $\vec{x} = \sum_{c=1}^n x_c \vec{e}_c$ is a vector of $(F_{v_{j_\delta}}^+)^n$ and, more precisely, a point of $g_L^{(n)}[j_\delta, m_{j_\delta}]$.

The sum of all bifunctions on the conjugacy class representatives of $G^{(n)}(F_{\bar{v}}^{+,T} \times F_v^{+,T})$ is:

$$\begin{aligned} \phi_{G_R^T}^{(n)}(x_{g_R^T}) \otimes \phi_{G_L^T}^{(n)}(x_{g_L^T}) &= \bigoplus_{j_\delta=1}^r \bigoplus_{m_{j_\delta}} (\phi_{G_R^T}^{(n)}(x_{g_{j_\delta R}^T}) \otimes \phi_{G_L^T}^{(n)}(x_{g_{j_\delta L}^T})) \\ &= \bigoplus_{j_\delta} \bigoplus_{m_{j_\delta}} (\lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{-2\pi i j_\delta x} \otimes \lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{2\pi i j_\delta x}) \\ &= \text{ELLIP}_R(n, j_\delta, m_{j_\delta}) \otimes \text{ELLIP}_L(n, j_\delta, m_{j_\delta}) \end{aligned}$$

where

$$\text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) = \text{ELLIP}_R(n, j_\delta, m_{j_\delta}) \otimes \text{ELLIP}_L(n, j_\delta, m_{j_\delta})$$

constitutes the n -dimensional irreducible elliptic (and cuspidal) representation $\text{Irr ELLIP}(\text{GL}_n(\mathbb{A}_{F_v^+, T} \times \mathbb{A}_{F_v^+, T}))$ of $\text{GL}_n(F_v^{+, T} \times F_v^{+, T})$, where $\mathbb{A}_{F_v^+, T}$ and $\mathbb{A}_{F_v^+, T}$ are toroidal adèle semirings according to section 1.1.

4.1.3 The n -dimensional real irreducible global correspondence of Langlands

The global correspondence considered here is thus given by:

$$\begin{array}{ccc} \text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \longrightarrow & \text{Irr ELLIP}_{R \times L}(\text{GL}_n(\mathbb{A}_{F_v^+, T} \times \mathbb{A}_{F_v^+, T})) \\ \parallel & & \parallel \\ G^{(n)}(F_{v_\oplus}^{+, T} \times F_{v_\oplus}^{+, T}) & \longrightarrow & \text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) \end{array}$$

where:

- $\text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ is the sum of the products, right by left, of the equivalence classes of the irreducible n -dimensional Weil-Deligne representation of the bilinear global Weil group $(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ given by the algebraic bilinear real semigroup $G^{(n)}(F_{v_\oplus}^+ \times F_{v_\oplus}^+)$;
- $\text{Irr ELLIP}_{R \times L}(\text{GL}_n(\mathbb{A}_{F_v^+, T} \times \mathbb{A}_{F_v^+, T}))$ is the sum of the products, right by left, of the equivalence classes of the irreducible elliptic representation of $\text{GL}_n(F_v^{+, T} \times F_v^{+, T})$ given by the n -dimensional global elliptic bisemimodule $\text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta})$.

4.1.4 The singularization of the bisemisheaf on $G^{(n)}(F_v^+ \times F_v^+)$

As we are concerned with the problem of (degenerate) singularities in the global program of Langlands, we shall take into account the set of real-valued differentiable bifunctions $\{\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta R}}) \otimes \phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta L}})\}$ on the corresponding conjugacy class representatives $\{g_{R \times L}^{(n)}[j_\delta, m_{j_\delta}]\}$ of the bilinear real algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$, as mentioned in section 1.6. And, more precisely, we shall work with the bisemisheaf of rings $\theta_{G_{R \times L}^{(n)}} = \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$, whose (bi)sections are the differentiable bifunctions $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta R}}) \otimes \phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta L}})$ (see section 1.7).

Let $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) be the semisheaf of left (resp. right) differentiable functions $\phi_{G_{j_L}}^{(n)}(x_{g_{j_\delta L}})$ (resp. $\phi_{G_{j_R}}^{(n)}(x_{g_{j_\delta R}})$), rewritten in a condensed form according to $\phi_{j_\delta}(x_L)$ (resp. $\phi_{j_\delta}(x_R)$).

The singularization of $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) is given by the contracting surjective morphism (see section 2.1):

$$\bar{\rho}_{G_L} : \theta_{G_L^{(n)}} \longrightarrow \theta_{G_L^{(n)}}^* \quad (\text{resp. } \bar{\rho}_{G_R} : \theta_{G_R^{(n)}} \longrightarrow \theta_{G_R^{(n)}}^*)$$

where $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) denotes the left (resp. right) singular semisheaf whose sections $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) are endowed with germs $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) having degenerate singularities of corank 1.

The corresponding singular bisemisheaf is $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ degenerate from $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ under the contracting surjective morphism:

$$\bar{\rho}_{G_R} \times \bar{\rho}_{G_L} : \theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}} \longrightarrow \theta_{G_R^{(n)}}^* \times \theta_{G_L^{(n)}}^* .$$

4.1.5 The versal deformation of the singular bisemisheaf $(\theta_{G_R^{(n)}}^* \times \theta_{G_L^{(n)}}^*)$

Let

$$D_{S_L} : \theta_{G_L^{(n)}}^* \times \theta_{S_L} \longrightarrow \theta_{G_L^{(n)}}^* \quad (\text{resp. } D_{S_R} : \theta_{G_R^{(n)}}^* \times \theta_{S_R} \longrightarrow \theta_{G_R^{(n)}}^*)$$

be the versal deformation of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$). It is a contracting fibre bundle whose fibre $\theta_{S_L} = \{\theta^1(\omega_L^1), \dots, \theta^1(\omega_L^i), \dots, \theta^1(\omega_L^s)\}$ (resp. $\theta_{S_R} = \{\theta^1(\omega_R^1), \dots, \theta^1(\omega_R^i), \dots, \theta^1(\omega_R^s)\}$) is the family of (semi)sheaves of the left (resp. right) base S_L (resp. S_R) of the versal deformation.

As developed in section 2.1, the versal deformation of a degenerate singularity of corank 1 and codimension s on a section $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) is yielded by a sequence of s contracting morphisms extending the corresponding sequence of contracting surjective morphisms of singularization.

4.1.6 The spreading-out isomorphism

It consists in a blow-up of the versal deformation $\theta_{G_L^{(n)}}^* \times \theta_{S_L}$ (resp. $\theta_{G_R^{(n)}}^* \times \theta_{S_R}$) of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$).

This blow-up is maximal if the spreading-out isomorphism $SOT_L^{\max} = (\tau_{V_{\omega_L}} \circ \pi_{s_L}^{\max})$ (resp. $SOT_R^{\max} = (\tau_{V_{\omega_R}} \circ \pi_{s_R}^{\max})$) is the inverse of the versal deformation:

$$SOT_L^{\max} = (D_{S_L})^{-1} \quad (\text{resp. } SOT_R^{\max} = (D_{S_R})^{-1}).$$

It is then given by the map:

$$\begin{aligned} SOT_L^{\max} : \theta_{G_L^{(n)}}^* \times \theta_{S_L} &\longrightarrow \theta_{G_L^{(n)}}^* \cup \theta_{S_L} \\ (\text{resp. } SOT_R^{\max} : \theta_{G_R^{(n)}}^* \times \theta_{S_R} &\longrightarrow \theta_{G_R^{(n)}}^* \cup \theta_{S_R}) \end{aligned}$$

projecting the family of sheaves of the left (resp. right) base S_L (resp. S_R) of the versal deformation in the vertical tangent space $T_{V_{\omega_L}}$ (resp. $T_{V_{\omega_R}}$) (see proposition 3.1.16 and corollary 3.1.17).

Let $\theta_{SOT(1)_L}^*$ (resp. $\theta_{SOT(1)_R}^*$) be the family θ_{S_L} (resp. θ_{S_R}) of disconnected base semisheaves having been glued together according to section 3.1.20:

The semisheaf $\theta_{SOT(1)_L}^* \simeq \theta_{S_L}$ (resp. $\theta_{SOT(1)_R}^* \simeq \theta_{S_R}$) covers partially the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) but can be affected by singularities in its sections... which can also undergo a versal deformation.

For the simplicity, we shall consider that we are only confronted with the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) and its first cover $\theta_{SOT(1)_L}^*$ (resp. $\theta_{SOT(1)_R}^*$), having possible singularities.

4.1.7 Contracting morphisms of singularization

The bisemisheaf $\theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}}$, being defined on the bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$, constitutes an n -dimensional irreducible real representation $\text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}^{(n)}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ of the bilinear global Weil group.

The fact of considering contracting morphisms of singularization leads to the transformation of $\theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}}$ into:

$$\begin{aligned} \theta_{G_R^{(n)}} \times \theta_{G_L^{(n)}} &\xrightarrow{\bar{\rho}_{G_R} \times \bar{\rho}_{G_L}} \theta_{G_R^{(n)}}^* \times \theta_{G_L^{(n)}}^* \\ &\xrightarrow{D_{S_R} \times D_{S_L}} (\theta_{G_R^{(n)}}^* \times \theta_{S_R}) \otimes (\theta_{G_L^{(n)}}^* \times \theta_{S_L}) \\ &\xrightarrow{SOT_R^{\max} \times SOT_L^{\max}} (\theta_{G_R^{(n)}}^* \cup \theta_{SOT(1)_R}^*) \otimes (\theta_{G_L^{(n)}}^* \cup \theta_{SOT(1)_L}^*) \end{aligned}$$

where:

- $\bar{\rho}_{G_R} \times \bar{\rho}_{G_L}$ is the contracting morphism of singularization;
- $D_{S_R} \times D_{S_L}$ is the contracting morphism of versal deformation;
- $SOT_R^{\max} \times SOT_L^{\max}$ is the (contracting) blow-up of the versal deformation.

But, the bisemisheaves $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and $(\theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^*)$, being affected by singularities, cannot be endowed with a cuspidal representation.

To reach this objective, it is necessary to:

- 1) desingularize these bisemisheaves;
- 2) submit them to a toroidal compactification.

This is the aim of the next section.

4.2 Langlands global correspondences despite of degenerate singularities

4.2.1 Desingularizing the bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$

The desingularization of the semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) corresponds to the classical monoidal transformations applied to the singularities on the sections $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) of $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$).

A desingularization of $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) is described succinctly in proposition 2.1.3 and corresponds to the inverse morphism of a singularization developed in section 2.1.

More concretely, if we want to desingularize a germ $\phi_{j_\delta}(\omega_L)$ (resp. $\phi_{j_\delta}(\omega_R)$) on $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$), given by the degenerate singularity $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) of corank 1 and codimensions s , we have to consider the following sequence of expanding morphisms:

$$\begin{aligned} \overline{\rho}_L^{(\text{desing})} : \quad \omega_L^{s+2} &\xrightarrow{\overline{\rho}_L^{(-1)(s+2)}} \omega_L^{s+1} \cup D_L^{(s+1)} \xrightarrow{\overline{\rho}_L^{(-1)(s+1)}} \omega_L^s \cup D_L^s \longrightarrow \dots \\ &\xrightarrow{\overline{\rho}_L^{(-1)(1)}} \omega_L \cup D_L^{(1)} \\ (\text{resp. } \overline{\rho}_R^{(\text{desing})} : \quad \omega_R^{s+2} &\xrightarrow{\overline{\rho}_R^{(-1)(s+2)}} \omega_R^{s+1} \cup D_R^{(s+1)} \xrightarrow{\overline{\rho}_R^{(-1)(s+1)}} \omega_R^s \cup D_R^s \longrightarrow \dots \\ &\xrightarrow{\overline{\rho}_R^{(-1)(1)}} \omega_R \cup D_R^{(1)}) \end{aligned}$$

where:

- 1) the expanding morphism of desingularization

$$\overline{\rho}_L^{(-1)(s+1)} : \quad \omega_L^{s+1} \longrightarrow \omega_L^s \cup D_L^s \quad (\text{resp. } \overline{\rho}_R^{(-1)(s+1)} : \quad \omega_R^{s+1} \longrightarrow \omega_R^s \cup D_R^s)$$

is a projective morphism, disconnecting the divisor D_L^s (resp. D_R^s) from the singular sublocus

$$\Sigma_L^{(s+1)} = \omega_L^{s+1} \quad (\text{resp. } \Sigma_R^{(s+1)} = \omega_R^{s+1}).$$

$$2) \quad \overline{\rho}_L^{(\text{desing})} : \quad \phi_{j_\delta}^*(x_L) \longrightarrow \phi_{j_\delta}(x_L) \cup (D_L^{(s+1)}, D_L^{(s)}, \dots, D_L^{(1)})$$

$$(\text{resp. } \overline{\rho}_R^{(\text{desing})} : \quad \phi_{j_\delta}^*(x_R) \longrightarrow \phi_{j_\delta}(x_R) \cup (D_R^{(s+1)}, D_R^{(s)}, \dots, D_R^{(1)}))$$

is the desingularization of $y_L = \omega_L^{s+2}$ (resp. $y_R = \omega_R^{s+2}$) on $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) disconnecting (by projection) the set of divisors $(D_L^{(s+1)}, D_L^{(s)}, \dots, D_L^{(1)})$ (resp. $(D_R^{(s+1)}, D_R^{(s)}, \dots, D_R^{(1)})$) which generate a real projective subscheme of dimension $(s-1)$.

As a result, $\bar{\phi}_{j_\delta}(x_L)$ (resp. $\bar{\phi}_{j_\delta}(x_R)$) becomes a smooth section of $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$).

- 3) the expanding morphism of desingularization $\bar{\rho}_L^{(\text{desing})}$ (resp. $\bar{\rho}_R^{(\text{desing})}$) is an isomorphism outside the singular locus $\Sigma_L = \omega_L^{s+2}$ (resp. $\Sigma_R = \omega_R^{s+2}$) according to:

$$\begin{aligned} \bar{\rho}_L^{(-1)\text{is}} : \quad \phi_{j_\delta}(x_L) \setminus (\Sigma_L) &\longrightarrow \bar{\phi}_{j_\delta}(x_L) \setminus (\bar{\rho}_L^{(-1)}(\Sigma_L)) \\ (\text{resp. } \bar{\rho}_R^{(-1)\text{is}} : \quad \phi_{j_\delta}(x_R) \setminus (\Sigma_R) &\longrightarrow \bar{\phi}_{j_\delta}(x_R) \setminus (\bar{\rho}_R^{(-1)}(\Sigma_R)) \end{aligned}$$

This desingularization process is exactly the inverse of the singularization developed in proposition 2.1.10.

The desingularization or resolution of singularities on all the sections $\phi_{j_\delta}^*(x_L)$ (resp. $\phi_{j_\delta}^*(x_R)$) of the singular semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) is given by the set of expanding morphisms:

$$\bar{\rho}_{G_L}^{(\text{desing})} : \theta_{G_L^{(n)}}^* \longrightarrow \theta_{G_L^{(n)}} \quad (\text{resp. } \bar{\rho}_{G_R}^{(\text{desing})} : \theta_{G_R^{(n)}}^* \longrightarrow \theta_{G_R^{(n)}})$$

in such a way that:

$$\bar{\rho}_{G_L}^{(\text{desing})} = \bar{\rho}_{G_L}^{-1} \quad (\text{resp. } \bar{\rho}_{G_R}^{(\text{desing})} = \bar{\rho}_{G_R}^{-1})$$

where $\bar{\rho}_{G_L}$ (resp. $\bar{\rho}_{G_R}$) denotes the set of contracting morphisms of singularization of $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$).

And, the resolution of singularities of the singular semisheaf $\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*$ is given by:

$$\bar{\rho}_{G_R}^{(\text{desing})} \times \bar{\rho}_{G_L}^{(\text{desing})} : \theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^* \longrightarrow \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}.$$

4.2.2 Resolution of singularities of the covering bisemisheaf $\theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^*$

As the sections of the semisheaf $\theta_{G_L^{(n)}}^*$ (resp. $\theta_{G_R^{(n)}}^*$) are endowed with singularities of corank 1 and codimension s , the semisheaf $\theta_{SOT(1)_L}^*$ (resp. $\theta_{SOT(1)_R}^*$) (being the family of “ s ” base semisheaves of the versal deformation having been glued together) can be affected on its sections by singularities of corank 1 and maximal codimensions equal to $(s-2)$ according to section 3.1 and proposition 2.2.8.

So, a resolution of singularities of this covering bisemisheaf $\theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^*$ must be envisaged as it was done for the bisemisheaf $\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*$.

The resolution of singularities of $\theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^*$ will then be given by the morphism:

$$\overline{\rho}_{SOT(1)_R}^{(\text{desing})} \times \overline{\rho}_{SOT(1)_L}^{(\text{desing})} : \theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^* \longrightarrow \theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$$

where $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ denotes the free corresponding bisemisheaf.

4.2.3 Global holomorphic representation of $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$

As the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ was desingularized from its corresponding singular equivalent $\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*$, a n -dimensional irreducible global holomorphic representation can be worked out for it, as it was developed in section 3.1 of [Pie1]. We shall recall it briefly.

The sections of the semisheaf $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) are the complex-valued differentiable functions (resp. cofunctions):

$$\begin{aligned} f_{v_{j_\delta}, m_{j_\delta}}(z^{j_\delta}) : g_L^{(n)}[j_\delta, m_{j_\delta}] &\longrightarrow F_{\omega_j} \\ (\text{resp. } \overline{f}_{\overline{v}_{j_\delta}, m_{j_\delta}}(z^{*j_\delta}) : g_L^{(n)}[j_\delta, m_{j_\delta}] &\longrightarrow F_{\overline{\omega}_j}) \end{aligned}$$

on the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) of $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\overline{v}}^+)$) on which $\theta_{G_L^{(n)}}$ (resp. $\theta_{G_R^{(n)}}$) is defined.

z^{j_δ} (resp. z^{*j_δ}) are the coordinate functions on $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) with respect to the charts:

$$\begin{aligned} c_{j_\delta, m_{j_\delta}} z^{j_\delta} : g_L^{(n)}[j_\delta, m_{j_\delta}] &\longrightarrow g_L^{(n)}[j, m_j] \\ (\text{resp. } c_{j_\delta, m_{j_\delta}}^* z^{*j_\delta} : g_R^{(n)}[j_\delta, m_{j_\delta}] &\longrightarrow g_R^{(n)}[j, m_j]) \end{aligned}$$

where $g_L^{(n)}[j, m_j]$ (resp. $g_R^{(n)}[j, m_j]$) are the corresponding complex conjugacy class representatives.

If the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) are glued together, then, a Laurent polynomial corresponding to the mapping:

$$f_v(z) : G^{(n)}(F_v^+) \longrightarrow F_\omega \quad (\text{resp. } \overline{f}_{\overline{v}}(z^*) : G^{(n)}(F_{\overline{v}}^+) \longrightarrow F_{\overline{\omega}})$$

is given, on $G^{(n)}(F_{v_\oplus}^+)$ (resp. $G^{(n)}(F_{\overline{v}_\oplus}^+)$), by:

$$\begin{aligned} f_v(z) &= \sum_{j_\delta=1}^r \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}} z^{j_\delta}, \quad 1 \leq j_\delta \leq r \leq \infty \\ (\text{resp. } \overline{f}_{\overline{v}}(z^*) &= \sum_{j_\delta=1}^r \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}}^* z^{*j_\delta} .) \end{aligned}$$

where: $F_{v_\oplus}^+ = \bigoplus_{j_\delta=1}^r \bigoplus_{m_{j_\delta}} F_{v_{j_\delta}, m_{j_\delta}}^+$ (resp. $F_{\overline{v}}^+ = \bigoplus_{j_\delta=1}^r \bigoplus_{m_{j_\delta}} F_{\overline{v}_{j_\delta}, m_{j_\delta}}^+$).

Note that the Laurent polynomial $f_v(z)$ (resp. $f_{\bar{v}}(z^*)$) is the sum of the Laurent monomials $f_{v_{j_\delta}}(z^{j_\delta})$ (resp. $f_{\bar{v}_{j_\delta}}(z^{*j_\delta})$) on $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$):

$$f_v(z) = \sum_{j_\delta} \sum_{m_{j_\delta}} f_{v_{j_\delta}, m_{j_\delta}}(z^{j_\delta}) \quad (\text{resp. } f_{\bar{v}}(z^*) = \sum_{j_\delta} \sum_{m_{j_\delta}} f_{\bar{v}_{j_\delta}, m_{j_\delta}}(z^{*j_\delta})).$$

So, on $G^{(n)}(F_{v_\oplus}^+)$ (resp. $G^{(n)}(F_{\bar{v}_\oplus}^+)$), the function $f_v(z)$ (resp. $f_{\bar{v}}(z^*)$), defined in a neighbourhood of a point z_0 (resp. z_0^*) of \mathbb{C}^n , is holomorphic at z_0 (resp. z_0^*) if we have the multiple power series development:

$$f_v(z) = \sum_{j_\delta=1}^{\infty} \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}} (z_1 - z_{01})^{j_\delta} \cdots (z_n - z_{0n})^{j_\delta}$$

$$(\text{resp. } f_{\bar{v}}(z^*) = \sum_{j_\delta=1}^{\infty} \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}}^* (z_1^* - z_{01}^*)^{j_\delta} \cdots (z_n^* - z_{0n}^*)^{j_\delta})$$

where:

- $z_1, z_{01}, \dots, z_n, z_{0n}$ are complex functions of one real variable;
- $z_i : F_{v_{i\sigma}}^+ \longrightarrow F_{\omega_i^1}$
- $c_{j_\delta, m_{j_\delta}}$ (resp. $c_{j_\delta, m_{j_\delta}}^*$) is in one-to-one correspondence with the product of the square roots of the eigenvalues of the (j_δ, m_{j_δ}) -th coset representative $U_{j_\delta, m_{j_\delta R}} \times U_{j_\delta, m_{j_\delta L}}$ of the product $T_R(n; t) \otimes T_L(n; t)$ of the Hecke operators.

And, the global holomorphic representation $\text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ is given by the morphism:

$$\text{Irr hol}_{\theta_{G_R \times L}}^{(n)} : \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z)$$

where $f_{\bar{v}}(z^*) \otimes f_v(z)$ is the holomorphic bifunction obtained by gluing together and adding the bisections of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$.

4.2.4 Holomorphic representation of the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$

The singularities of the covering bisemisheaf $\theta_{SOT(1)_R}^* \otimes \theta_{SOT(1)_L}^*$ having been resolved, the free bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ can be endowed with a holomorphic representation as it was done in section 4.2.3 for $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$.

Let $g_{SOT(1)_L}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}]$ (resp. $g_{SOT(1)_R}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}]$) denote the conjugacy class representative of the algebraic semigroup $G^{(n)}(F_{v_{\text{cov}}}^+)$ (resp. $G^{(n)}(F_{\bar{v}_{\text{cov}}}^+)$) covering $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+)$) where $F_{v_{\text{cov}}}^+$ denotes the set of covering completions in such a way that

the completions $F_{v_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}}^+}$ are characterized by ranks $r_{j_{\delta(\text{cov})}} = j_{\delta(\text{cov})} \cdot N$ inferior or equal to corresponding ranks $r_{j_{\delta}} = j_{\delta} \cdot N$ of $F_{v_{j_{\delta}, m_{j_{\delta}}}}^+$: so, $j_{\delta(\text{cov})} \leq j_{\delta}$.

The sections of the semisheaf $\theta_{SOT(1)_L}^*$ (resp. $\theta_{SOT(1)_R}^*$) are the complex-valued differentiable functions (resp. cofunctions):

$$\begin{aligned} f_{v_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}} (z^{j_{\delta(\text{cov})}}) : g_{SOT(1)_L}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}] &\longrightarrow F_{\omega_{j_{\delta(\text{cov})}}} \\ (\text{resp. } f_{\bar{v}_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}} (z^{*j_{\delta(\text{cov})}}) : g_{SOT(1)_R}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}] &\longrightarrow F_{\bar{\omega}_{j_{\delta(\text{cov})}}}) \end{aligned}$$

If the conjugacy class representatives $g_{SOT(1)_L}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}]$ (resp. $g_{SOT(1)_R}^{(n)}[j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}]$) are glued together, then the Laurent polynomial

$$\begin{aligned} f_{v_{\text{cov}}}(z_{\text{cov}}) &= \sum_{j_{\delta(\text{cov})}=1}^r \sum_{m_{j_{\delta(\text{cov})}}} c_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}} z^{j_{\delta(\text{cov})}}, \quad 1 \leq j_{\delta(\text{cov})} \leq r \leq \infty \\ (\text{resp. } f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) &= \sum_{j_{\delta(\text{cov})}=1}^r \sum_{m_{j_{\delta(\text{cov})}}} c_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}}^* z^{*j_{\delta(\text{cov})}}) \end{aligned}$$

can be defined on $G^{(n)}(F_{v_{\text{cov}}}^+)$ (resp. $G^{(n)}(F_{\bar{v}_{\text{cov}}}^+)$) where $c_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}}$ is in one-to-one correspondence with the product of the square roots of the eigenvalues of the $(j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}})$ -th coset representative of the product of the Hecke operators.

And, the corresponding holomorphic function (resp. cofunction) will be given by the multiple power series development:

$$\begin{aligned} f_{v_{\text{cov}}}(z_{\text{cov}}) &= \sum_{j_{\delta(\text{cov})}=1}^r \sum_{m_{j_{\delta(\text{cov})}}} c_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}} (z_1 - z_{01})^{j_{\delta(\text{cov})}} \cdots (z_n - z_{0n})^{j_{\delta(\text{cov})}} \\ (\text{resp. } f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) &= \sum_{j_{\delta(\text{cov})}=1}^r \sum_{m_{j_{\delta(\text{cov})}}} c_{j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}}^* (z_1^* - z_{01}^*)^{j_{\delta(\text{cov})}} \cdots (z_n^* - z_{0n}^*)^{j_{\delta(\text{cov})}}). \end{aligned}$$

The global homomorphic representation $\text{Irr hol}^{(n)}(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$ of the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ is given by the morphism:

$$\text{Irr hol}_{\theta_{SOT(1)_R \times L}}^{(n)} : \theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L} \longrightarrow f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}}).$$

4.2.5 Covering n -dimensional representation of Weil groups

As the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ is defined on the covering algebraic bilinear semigroup $G^{(n)}(F_{\bar{v}_{\text{cov}}}^+ \times F_{v_{\text{cov}}}^+)$, a n -dimensional irreducible real representation $\text{Irr Rep}_{F_R^{\text{cov}} \times F_L^{\text{cov}}}^{(n)}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab})$ of the bilinear global Weil group $(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab})$ must correspond to it.

The global Weil groups $W_{F_R^{\text{cov}}}^{ab}$ and $W_{F_L^{\text{cov}}}^{ab}$ may be defined as in section 4.1.1 by:

$$W_{F_R^{\text{cov}}}^{ab} = \bigoplus_{j_{\delta(\text{cov})}} \bigoplus_{m_{j_{\delta(\text{cov})}}} \text{Gal}(\tilde{F}_{\bar{v}_{j_{\delta(\text{cov})}}, m_{j_{\delta(\text{cov})}}}^+ / F^0)$$

$$W_{F_L^{\text{cov}}}^{ab} = \bigoplus_{j_{\delta(\text{cov})}} \bigoplus_{m_{j_{\delta(\text{cov})}}} \text{Gal}(\tilde{F}_{v_{j_{\delta(\text{cov})}}, m_{j_{\delta(\text{cov})}}}^+ / F^0)$$

where $\tilde{F}_{v_{j_{\delta(\text{cov})}}, m_{j_{\delta(\text{cov})}}}^+$ is the ramified Galois extension corresponding to the completion $F_{v_{j_{\delta(\text{cov})}}, m_{j_{\delta(\text{cov})}}}^+$.

4.2.6 Proposition

On the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ affected by degenerate singularities, the following global holomorphic correspondences exist:

$$\begin{array}{ccc}
 \text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \xrightarrow{\quad} & \text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}) \\
 \parallel & & \parallel \\
 \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} & \xrightarrow{\quad} & f_{\bar{v}}(z^*) \otimes f_v(z) \\
 & \searrow \bar{\rho}_{G_R} \times \bar{\rho}_{G_L} & \\
 & \theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^* & \\
 & \downarrow D_{S_R} \times D_{S_L} & \\
 & (\theta_{G_R^{(n)}}^* \times \theta_{S_R}) \otimes (\theta_{G_L^{(n)}}^* \times \theta_{S_L}) & \\
 & \swarrow SOT_R^{\max} \times SOT_L^{\max} & \\
 (\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*) \cup (\theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^*) & & \\
 & \swarrow \bar{\rho}_{SOT(1)R}^{(\text{desing})} \times \bar{\rho}_{SOT(1)L}^{(\text{desing})} & \\
 \theta_{SOT(1)R} \otimes \theta_{SOT(1)L} & \xrightarrow{\quad} & f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}}) \\
 \parallel & & \parallel \\
 \text{Irr Rep}_{W_{F_R^{\text{cov}} \times L}^{(n)}}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab}) & \xrightarrow{\quad} & \text{Irr hol}^{(n)}(\theta_{SOT(1)R} \otimes \theta_{SOT(1)L})
 \end{array}$$

where:

- 1) $\text{Irr Rep}_{W_{F_R^+ \times F_L^+}}^{(n)}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) \rightarrow \text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$ is the global holomorphic correspondence on the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ submitted consecutively to:
 - a) the singularization morphism $\bar{\rho}_{G_R} \times \bar{\rho}_{G_L}$;
 - b) the versal deformation $D_{S_R} \times D_{S_L}$;
 - c) the blow- up $SOT_R^{\max} \times SOT_L^{\max}$ of the versal deformation;
 - d) the desingularization $\bar{\rho}_{G_R}^{(\text{desing})} \times \bar{\rho}_{G_L}^{(\text{desing})}$.
- 2) $\text{Irr Rep}_{W_{F_R^{\text{cov}} \times F_L^{\text{cov}}}}^{(n)}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab}) \rightarrow \text{Irr hol}^{(n)}(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$ is the global holomorphic correspondence on the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ generated by the versal deformation $D_{S_R} \times D_{S_L}$ followed by the spreading-out isomorphism $SOT_R^{\max} \times SOT_L^{\max}$ of the singular bisemisheaf $\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*$.

Proof. This diagram proceeds from the preceding developments. It then results that the blow-up of the versal deformation of the singular bisemisheaf $\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*$ generates the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ from which a new global holomorphic correspondence can be established. ■

4.2.7 Toroidal compactification

In order to get a possible automorphic representation of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ on the bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$ and of the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ on the covering bilinear algebraic semigroup $G^{(n)}(F_{v_{\text{cov}}}^+ \times F_{v_{\text{cov}}}^+)$, a toroidal compactification of these bilinear algebraic semigroups $G^{(n)}(F_v^+ \times F_v^+)$ and $G^{(n)}(F_{v_{\text{cov}}}^+ \times F_{v_{\text{cov}}}^+)$ must be realized.

According to propositions 3.2.2 and 3.2.3, the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}] \in G^{(n)}(F_v^+)$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}] \in G^{(n)}(F_v^+)$) decompose into $(j_\delta)^n$ completions of rank N .

The toroidal compactification of the linear algebraic semigroup $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_v^+)$) can be carried out by considering the horizontal rotational tangent bundle (see section 3.2.4):

$$\begin{aligned} \tau_{G_L^{(n)}(F_v^+)} : T(G_L^{(n)}(F_v^{+,T})) &\longrightarrow G_L^{(n)}(F_v^{+,T}) \\ \text{(resp. } \tau_{G_R^{(n)}(F_v^+)} : T(G_R^{(n)}(F_v^{+,T})) &\longrightarrow G_R^{(n)}(F_v^{+,T}) \end{aligned}$$

whose total space $T(G_L^{(n)}(F_v^{+,T}))$ (resp. $T(G_R^{(n)}(F_{\bar{v}}^{+,T}))$) is a projective linear algebraic semigroup $PG_L^{(n)}(F_v^{+,T})$ (resp. $PG_R^{(n)}(F_{\bar{v}}^{+,T})$).

The sections $g_{T_L}^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_{T_R}^{(n)}[j_\delta, m_{j_\delta}]$) of the total space $T(G_L^{(n)}(F_v^{+,T}))$ (resp. $T(G_R^{(n)}(F_{\bar{v}}^{+,T}))$) are n -dimensional real semitori whose one-dimensional fibres are semi-circles obtained by toroidal deformation of the completions of rank $N \cdot j_\delta$ of $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) under the action of the horizontal tangent bundle $\tau_{G_L^{(n)}(F_v^{+,T})}$ (resp. $\tau_{G_R^{(n)}(F_{\bar{v}}^{+,T})}$).

Remark that this kind of toroidal compactification is in one-to-one correspondence with the toroidal compactification of the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) given in terms of the projective emergent isomorphism γ_L^c (resp. γ_R^c) introduced in sections 3.2 and 3.3 of [Pie1].

The toroidal compactification of the covering algebraic semigroup $G_L^{(n)}(F_{v_{\text{cov}}}^+)$ (resp. $G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^+)$) can be performed similarly, i.e. by considering the horizontal rotational tangent bundle:

$$\begin{aligned} \tau_{G_L^{(n)}(F_{v_{\text{cov}}}^{+,T})} : T(G_L^{(n)}(F_{v_{\text{cov}}}^{+,T})) &\longrightarrow G_L^{(n)}(F_{v_{\text{cov}}}^{+,T}) \\ (\text{resp. } \tau_{G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T})} : T(G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T})) &\longrightarrow G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T})) \end{aligned}$$

in such a way that:

- 1) the total space of $\tau_{G_L^{(n)}(F_{v_{\text{cov}}}^{+,T})}$ (resp. $\tau_{G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T})}$) is given by $T(G_L^{(n)}(F_{v_{\text{cov}}}^{+,T}))$ (resp. $T(G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T}))$).
- 2) the sections of the total space $T(G_L^{(n)}(F_{v_{\text{cov}}}^{+,T}))$ (resp. $T(G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T}))$) covering the sections $g_{T_L}^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_{T_R}^{(n)}[j_\delta, m_{j_\delta}]$) are not necessarily complete n -dimensional real semitori because their one-dimensional fibres are in one-to-one correspondence with the completions of $G_L^{(n)}(F_{v_{\text{cov}}}^+)$ (resp. $G_R^{(n)}(F_{\bar{v}_{\text{cov}}}^+)$) having a rank $r_{j_{\delta_{\text{cov}}}} = j_{\delta_{\text{cov}}} \cdot N$ inferior or equal to the rank $r_{j_\delta} = j_\delta \cdot N$ of the completions of $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) (see section 4.2.4).

The bisemisheaf on the toroidal bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}}^{+,T} \times F_v^{+,T})$ will be noted $\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}$ and the bisemisheaf on the covering toroidal bilinear algebraic semigroup $G^{(n)}(F_{\bar{v}_{\text{cov}}}^{+,T} \times F_{v_{\text{cov}}}^{+,T})$ will be written $\theta_{G_{T_R}^{(n)}}^{\text{cov}} \otimes \theta_{G_{T_L}^{(n)}}^{\text{cov}}$.

4.2.8 Proposition

Let

$$\text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} : \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z))$$

be the global holomorphic representation of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ given by the holomorphic bifunction $f_{\bar{v}}(z^*) \otimes f_v(z)$ as introduced in section 4.2.3.

Then, $\tau^{\text{tor}}(\text{Irr hol}_{\theta_{G_{R \times L}^{(n)}}}^{(n)})$, denoting the toroidal compactification of the global holomorphic representation of $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$, generates the corresponding elliptic representation of the bisemisheaf $\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}$ according to:

$$\begin{array}{ccc} \text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}) : & \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} & \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z) \\ \downarrow \tau^{\text{tor}} \text{Irr hol}_{\theta_{G_{R \times L}^{(n)}}}^{(n)} & \downarrow & \downarrow \\ \text{Irr ELLIP}(\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}) : & \theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}} & \rightarrow \text{ELLIP}_{R \times L}(n, j_{\delta}, m_{j_{\delta}}) \end{array}$$

where:

$$\text{ELLIP}_{R \times L}(n, j_{\delta}, m_{j_{\delta}}) = \text{ELLIP}_R(n, j_{\delta}, m_{j_{\delta}}) \otimes \text{ELLIP}_L(n, j_{\delta}, m_{j_{\delta}})$$

being the global elliptic representation of $\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}$, is the product, right by left, of n -dimensional real global elliptic semimodules given by:

$$\begin{aligned} \text{ELLIP}_L &= \bigoplus_{j_{\delta}=1}^r \bigoplus_{m_{j_{\delta}}} \lambda^{\frac{1}{2}}(n, j_{\delta}, m_{j_{\delta}}) e^{2\pi i j_{\delta} x}, \\ \text{ELLIP}_R &= \bigoplus_{j_{\delta}=1}^r \bigoplus_{m_{j_{\delta}}} \lambda^{\frac{1}{2}}(n, j_{\delta}, m_{j_{\delta}}) e^{-2\pi i j_{\delta} x}, \end{aligned}$$

with:

- $\vec{x} = \sum_{c=1}^n x_c \vec{e}_c$ a vector of $(F_{v1}^+)^n$;
- $\lambda(n, j_{\delta}, m_{j_{\delta}}) = \prod_{c=1}^n \lambda_c(n, j_{\delta}, m_{j_{\delta}})$ according to section 4.1.2.

Proof. $\text{ELLIP}_{R \times L}(n, j_{\delta}, m_{j_{\delta}})$ is also the n -dimensional irreducible elliptic representation $\text{Irr ELLIP}_{R \times L}(\text{GL}_n(\mathbb{A}_{F_v^{+,T}} \times \mathbb{A}_{F_v^{+,T}}))$ of $\text{GL}_n(F_v^{+,T} \times F_v^{+,T})$ as developed in section 4.1.2.

In section 4.2.7, the toroidal compactification of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ into the bisemisheaf $\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}$ was realized.

It remains to prove that the holomorphic representation of $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$, given by the holomorphic bifunction $f_{\bar{v}}(z^*) \otimes f_v(z)$, is in one-to-one correspondence with the global elliptic bisemimodule $\text{ELLIP}_{R \times L}(n, j_{\delta}, m_{j_{\delta}})$ under the toroidal compactification of $f_{\bar{v}}(z^*) \otimes f_v(z)$.

This is evident, since $f_v(z)$ (resp. $f_{\bar{v}}(z^*)$) decomposes into:

$$f_v(z) = \sum_{j_\delta=1}^r \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}} z^{j_\delta} ,$$

$$\text{(resp. } f_{\bar{v}}(z^*) = \sum_{j_\delta=1}^r \sum_{m_{j_\delta}} c_{j_\delta, m_{j_\delta}}^* z^{*j_\delta} \text{) ,}$$

according to section 4.2.3, in such a way that:

- each term $c_{j_\delta, m_{j_\delta}} z^{j_\delta}$ (resp. $c_{j_\delta, m_{j_\delta}}^* z^{*j_\delta}$) is a complex-valued differentiable function on the conjugacy class representative $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$);
- the coefficient $c_{j_\delta, m_{j_\delta}}$ of $f_v(z)$ corresponds to the coefficient $\lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta})$ of $\text{ELLIP}_L(n, j_\delta, m_{j_\delta})$ according to sections 4.1.2 and 4.2.3.

The toroidal deformation of the conjugacy class representatives $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) into their toroidal equivalents $g_{T_L}^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_{T_R}^{(n)}[j_\delta, m_{j_\delta}]$) is such that the one-dimensional fibres of $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$), which are completions of rank $j_\delta \cdot N$, are transformed into semicircles, which are the fibres of $g_{T_L}^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_{T_R}^{(n)}[j_\delta, m_{j_\delta}]$) (see section 4.2.7).

So, the toroidal deformation of $g_L^{(n)}[j_\delta, m_{j_\delta}]$ (resp. $g_R^{(n)}[j_\delta, m_{j_\delta}]$) corresponds to the mapping:

$$\begin{aligned} \tau^{\text{tor}}[j_\delta, m_{j_\delta}] : g_L^{(n)}[j_\delta, m_{j_\delta}] &\longrightarrow g_{T_L}^{(n)}[j_\delta, m_{j_\delta}] \\ c_{j_\delta, m_{j_\delta}} z^{j_\delta} &\longrightarrow \lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{+2\pi i j_\delta x} , \end{aligned}$$

on the $[j_\delta, m_{j_\delta}]$ -th conjugacy class representative of $G^{(n)}(F_v^+)$, sending $c_{j_\delta, m_{j_\delta}} z^{j_\delta}$ into the n -dimensional real semitorus $\lambda^{\frac{1}{2}}(n, j_\delta, m_{j_\delta}) e^{2\pi i j_\delta x}$.

By adding the toroidal deformations on all conjugacy class representatives $g_R^{(n)}[j_\delta, m_{j_\delta}] \otimes g_L^{(n)}[j_\delta, m_{j_\delta}]$ of $G^{(n)}(F_{\bar{v}}^+ \times F_v^+)$, we get the searched one-to-one correspondence:

$$\tau^{\text{tor}}[F_{\bar{v}}^+ \times F_v^+] : \bigoplus_{j_\delta, m_{j_\delta}} (g_R^{(n)}[j_\delta, m_{j_\delta}] \otimes g_L^{(n)}[j_\delta, m_{j_\delta}]) \longrightarrow \bigoplus_{j_\delta, m_{j_\delta}} (g_{T_R}^{(n)}[j_\delta, m_{j_\delta}] \otimes g_{T_L}^{(n)}[j_\delta, m_{j_\delta}])$$

$$f_{\bar{v}}(z^*) \otimes f_v(z) \longrightarrow \text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) . \quad \blacksquare$$

4.2.9 Proposition

Let

$$\text{Irr hol}^{(n)}(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}) : \theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L} \longrightarrow f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}})$$

denote the global holomorphic representation of the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$ given by the holomorphic bifunction $f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}})$ introduced in section 4.2.4.

Then, $\tau^{\text{tor}}(\text{Irr hol}_{\theta_{SOT(1)_{R \times L}}^{(n)}})$, denoting the toroidal compactification of the global holomorphic representation of $(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$, generates the corresponding partial elliptic representation of the covering bisemisheaf $(\theta_{SOT(1)_{T_R}} \otimes \theta_{SOT(1)_{T_L}})$ according to:

$$\begin{array}{ccc} \text{Irr hol}^{(n)} \left(\begin{array}{c} \theta_{SOT(1)_R} \\ \otimes \theta_{SOT(1)_L} \end{array} \right) : & \begin{array}{c} \theta_{SOT(1)_R} \\ \otimes \theta_{SOT(1)_L} \end{array} & \longrightarrow f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}}) \\ \downarrow \tau^{\text{tor}} \text{Irr hol}_{\theta_{SOT(1)_{R \times L}}^{(n)}} & \downarrow & \downarrow \\ \text{Irr ELLIP}^{\text{part}} \left(\begin{array}{c} \theta_{SOT(1)_{T_R}} \\ \otimes \theta_{SOT(1)_{T_L}} \end{array} \right) : & \begin{array}{c} \theta_{SOT(1)_{T_R}} \\ \otimes \theta_{SOT(1)_{T_L}} \end{array} & \rightarrow \text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) \end{array}$$

where:

$$\text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) = \text{ELLIP}_R^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) \otimes \text{ELLIP}_L^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}})$$

is the product, right by left, of n -dimensional real partial global elliptic semimodules given by:

$$\text{ELLIP}_L^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) = \bigoplus_{j_{\delta_{(\text{cov})}}=1}^r \bigoplus_{m_{j_{\delta_{(\text{cov})}}}} \lambda^{\frac{1}{2}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) e^{i \cdot j_{\delta_{(\text{cov})}} \cdot x},$$

$$i \cdot j_{\delta_{\text{cov}}} \cdot x \leq \pi$$

$$\text{ELLIP}_R^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) = \bigoplus_{j_{\delta_{(\text{cov})}}=1}^r \bigoplus_{m_{j_{\delta_{(\text{cov})}}}} \lambda^{\frac{1}{2}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}}) e^{-i \cdot j_{\delta_{(\text{cov})}} \cdot x},$$

with:

- $x \in (F_{v_{(\text{cov})}}^+)^n$;
- $\lambda(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}})$ defined as in section 4.2.8.

Proof. This can be proved similarly as it was done in proposition 4.2.8: that is to say that the holomorphic representation of $(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$, given by the covering holomorphic bifunction $f_{\bar{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}})$, is in one-to-one correspondence with the global partial elliptic bisemimodule $\text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}})$.

The procedure is exactly the same as given in the proof of proposition 4.2.8. The only difference lies in the fact that $\text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}})$ is a global “**partial**” elliptic bi-semimodule. This results from the fact that, under the toroidal deformation of the conjugacy class representatives covering $g_L^{(n)}[j_{\delta}, m_{j_{\delta}}]$ (resp. $g_R^{(n)}[j_{\delta}, m_{j_{\delta}}]$), the one-dimensional fibres covering $g_L^{(n)}[j_{\delta}, m_{j_{\delta}}]$ (resp. $g_R^{(n)}[j_{\delta}, m_{j_{\delta}}]$) are completions of rank $j_{\delta(\text{cov})} \cdot N$ transformed into incomplete semicircles covering the semicircles of $g_{T_L}^{(n)}[j_{\delta}, m_{j_{\delta}}]$ (resp. $g_{T_R}^{(n)}[j_{\delta}, m_{j_{\delta}}]$) having ranks $j_{\delta} \cdot N \geq j_{\delta(\text{cov})} \cdot N$. ■

4.2.10 Proposition

On the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$, affected by degenerate singularities, the following global correspondences of Langlands, prolonging the global holomorphic correspondences of proposition 4.2.6, are:

$$\begin{array}{ccccc}
 \text{Irr Rep}_{W_{F_R^+ \times L}^{(n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \longrightarrow & \text{Irr hol}^{(n)}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}) & \xrightarrow{\tau^{\text{tor}} \text{Irr hol}^{(n)}_{\theta_{G_R \times L}}} & \text{Irr ELLIP}(\theta_{G_{T_R}^{(n)}} \otimes \theta_{G_{T_L}^{(n)}}) \\
 \parallel & & \parallel & & \parallel \\
 \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} & \longrightarrow & f_{\overline{v}}(z^*) \otimes f_v(z) & \longrightarrow & \text{ELLIP}_{R \times L}(n, j_{\delta}, m_{j_{\delta}}) \\
 & \searrow \overline{p}_{G_R} \times \overline{p}_{G_L} & & & \\
 & & \theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^* & & \\
 & & \downarrow D_{S_R} \times D_{S_L} & & \\
 & & (\theta_{G_R^{(n)}}^* \times \theta_{S_R}) \otimes (\theta_{G_L^{(n)}}^* \times \theta_{S_L}) & & \\
 & \swarrow SOT_R^{\text{max}} \times SOT_L^{\text{max}} & & & \\
 (\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*) & \cup & (\theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^*) & & \\
 & \swarrow \overline{p}_{SOT(1)R}^{(\text{desing})} \times \overline{p}_{SOT(1)L}^{(\text{desing})} & & & \\
 \theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^* & \longrightarrow & f_{\overline{v}_{\text{cov}}}(z_{\text{cov}}^*) \otimes f_{v_{\text{cov}}}(z_{\text{cov}}) & \longrightarrow & \text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}) \\
 \parallel & & \parallel & & \parallel \\
 \text{Irr Rep}_{W_{F_R^{\text{cov}} \times L}^{(n)}}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab}) & \longrightarrow & \text{Irr hol}^{(n)}(\theta_{SOT(1)R} \otimes \theta_{SOT(1)L}) & \longrightarrow & \text{Irr ELLIP}^{\text{part}}(\theta_{SOT(1)T_R} \otimes \theta_{SOT(1)T_L})
 \end{array}$$

Proof. This diagram shows how the Langlands global correspondence of section 3.4 of [Pie1] can be extended by considering degenerate singularities on the sections of the bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$ on the bilinear algebraic semigroup $G^{(n)}(F_v^+ \times F_v^+)$ constituting the irreducible n -dimensional Weil-Deligne representation $\text{Irr Rep}_{W_{F_R^+ \times F_L^+}}^{(n)}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ of the bilinear global Weil group $(W_{F_v^+}^{ab} \times W_{F_v^+}^{ab})$.

So, let:

$$\begin{array}{ccc} \text{Irr Rep}_{W_{F_R^+ \times F_L^+}}^{(n)}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \longrightarrow & \text{Irr ELLIP}(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}) \\ \parallel & & \parallel \\ \theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}} & \longrightarrow & \text{ELLIP}_{R \times L}(n, j_\delta, m_{j_\delta}) \end{array}$$

be the irreducible Langlands global correspondence of section 3.4 of [Pie1] applied to the free bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$.

The fact of considering:

- 1) a singularization $(\bar{\rho}_{G_R} \times \bar{\rho}_{G_L})$ of $(\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}})$;
- 2) a versal deformation $(D_{S_R} \times D_{S_L})$ of the singular bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$;
- 3) a spreading-out $(SOT_R^{\max} \times SOT_L^{\max})$ of the unfolded bisemisheaf $(\theta_{G_R^{(n)}}^* \times \theta_{S_R}^*) \otimes (\theta_{G_L^{(n)}}^* \otimes \theta_{S_L}^*)$ generating the singular bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and the covering singular bisemisheaf $(\theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^*)$;
- 4) a desingularization $(\bar{\rho}_{G_R}^{(\text{desing})} \times \bar{\rho}_{G_L}^{(\text{desing})})$ of the bisemisheaf $(\theta_{G_R^{(n)}}^* \otimes \theta_{G_L^{(n)}}^*)$ and a desingularization $(\bar{\rho}_{G_R}^{(\text{desing})} \times \bar{\rho}_{G_L}^{(\text{desing})})$ of the covering bisemisheaf $(\theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^*)$

allows to recover the desingularized original bisemisheaf $\theta_{G_R^{(n)}} \otimes \theta_{G_L^{(n)}}$, to which the above mentioned Langlands global correspondence can be reformulated, and an additional desingularized covering bisemisheaf $(\theta_{SOT(1)R}^* \otimes \theta_{SOT(1)L}^*)$ to which the following Langlands global correspondence can be stated:

$$\begin{array}{ccc} \text{Irr Rep}_{W_{F_R^{\text{cov}} \times F_L^{\text{cov}}}}^{(n)}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab}) & \longrightarrow & \text{Irr ELLIP}^{\text{part}}(\theta_{SOT(1)R} \otimes \theta_{SOT(1)L}) \\ \parallel & & \parallel \\ \theta_{SOT(1)R} \otimes \theta_{SOT(1)L} & \longrightarrow & \text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta(\text{cov})}, m_{j_{\delta(\text{cov})}}) \end{array}$$

where:

- $\text{Irr Rep}_{W_{F_R^{\text{cov}} \times F_L^{\text{cov}}}}^{(n)}(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab})$ is the sum of products, right by left, of the equivalence classes of the irreducible n -dimensional Weil-Deligne representation of the covering

bilinear global Weil group $(W_{F_R^{\text{cov}}}^{ab} \times W_{F_L^{\text{cov}}}^{ab})$ given by the covering bisemisheaf $\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L}$;

- $\text{Irr ELLIP}(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$ is the sum of the products, right by left, of the equivalence classes of the irreducible elliptic representation of the covering bisemisheaf $(\theta_{SOT(1)_R} \otimes \theta_{SOT(1)_L})$, given by the n -dimensional “partial” global elliptic bisemimodule $\text{ELLIP}_{R \times L}^{\text{part}}(n, j_{\delta_{(\text{cov})}}, m_{j_{\delta_{(\text{cov})}}})$. ■

4.2.11 Langlands REDUCIBLE global correspondences on bisemisheaves over reducible bilinear algebraic semigroups affected by degenerate singularities

The correspondences considered here can be worked out similarly as it was done in chapter 4 fo [Pie1], and, more particularly, in proposition 4.2.14.

So, on the basis of this proposition 4.2.14 of [Pie1], Langlands reducible global correspondences can be developed as it was done in the diagram of the preceding proposition 4.2.10 by taking into account that:

- 1) To each irreducible representation of a reducible bilinear algebraic semigroup, a diagram of Langlands global correspondence can be established as in proposition 4.2.10.
- 2) The bisemisheaves on the irreducible representations, decomposing the reducible representations of a bilinear algebraic semigroup, generate each one:
 - a) a Langlands global correspondence on the original desingularized bisemisheaf;
 - b) a Langlands global correspondence on the covering desingularized bisemisheaf, generated from the original bisemisheaf submitted to versal deformation and spreading-out morphism.

5 Langlands global correspondences over monodromy

In this chapter, the monodromy [Ebe1], [Ebe2], [Gro], [Gri] of (isolated) singularities on the (bisemi)sheaf of differentiable (bi)functions on the complex bilinear algebraic semigroup $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ is analysed and the Langlands global correspondences on the non singular fibres generated by monodromy are developed in the irreducible and reducible cases.

5.1 The monodromy of isolated singularities on irreducible complex semisheaves $\theta_{G_{L,R}^{(n)}}^{\mathbb{C}}$

5.1.1 Complex semisheaves on complex algebraic semigroups

As the Picard-Lefschetz theory is the complex analogue of Morse theory [Del3], [H-Z], our attention will be focused on singularities on the complex-valued differentiable functions $\phi_{G_L^{(n)}}^{(n)}(z_{g_L})$ (resp. cofunctions $\phi_{G_R^{(n)}}^{(n)}(z_{g_R})$) on the left (resp. right) complex linear algebraic semigroup $G^{(n)}(F_{\omega})$ (resp. $G^{(n)}(F_{\overline{\omega}})$), taking into account the inclusion $G^{(n)}(F_{\overline{v}}^+ \times F_v^+) \hookrightarrow G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ of the real bilinear algebraic semigroup $G^{(n)}(F_{\overline{v}}^+ \times F_v^+)$ into the corresponding complex equivalent $G^{(n)}(F_{\overline{\omega}} \times F_{\omega})$ according to section 1.9.

These complex-valued differentiable functions $\phi_{G_L^{(n)}}^{(n)}(z_{g_L})$ (resp. cofunctions $\phi_{G_R^{(n)}}^{(n)}(z_{g_R})$) are the sections $\phi_{G_{j_L}^{(n)}}^{(n)}(z_{g_{j_L}})$ (resp. $\phi_{G_{j_R}^{(n)}}^{(n)}(z_{g_{j_R}})$), $1 \leq j \leq r \leq \infty$, of a left (resp. right) semisheaf $\theta_{G_L^{(n)}}^{\mathbb{C}}$ (resp. $\theta_{G_R^{(n)}}^{\mathbb{C}}$) on the corresponding conjugacy class representatives $g_L^{(n)}[j, m_j]$ (resp. $g_R^{(n)}[j, m_j]$) of the complex linear algebraic semigroup $G^{(n)}(F_{\omega}) \equiv T_n(F_{\omega})$ (resp. $G^{(n)}(F_{\overline{\omega}}) \equiv T_n^t(F_{\overline{\omega}})$).

5.1.2 Monodromy in expanding phase

Remark that the singularisations and the versal deformations, developed in chapter 2, were envisaged in a contracting phase in the sense that:

- 1) the singularisation of a regular f -scheme is a contracting surjective morphism (see proposition 2.1.6).
- 2) the versal deformation of a semisheaf can be described as a contracting fibre bundle according to propositions 2.2.5 and 2.2.6.

And, the spreading-out, introduced as the blow-up of the versal deformation in section 3.1, also occurs naturally in a contracting phase, but the projective map of the tangent bundle on the disconnected base semisheaves has to be viewed as an expanding morphism. . . of blow-up (see section 3.1.15).

The contracting phase of a manifold reflects the fact that its submanifolds become closer and closer with respect to a fixed measure.

The monodromy, studied in this chapter, arises in an expanding phase as it will be justified in the following.

This expanding phase, reflecting the expansion of the submanifolds of a given manifold with respect to a fixed measure, is assumed to generate locally contracting surjective morphisms of singularisations as introduced in section 2.1.

5.1.3 Types of singularities

- Let $\phi_{G_{jL}^{(C)}}^{(n)}(z_{g_{jL}})$ (resp. $\phi_{G_{jR}^{(C)}}^{(n)}(z_{g_{jR}})$) be, as complex-valued differentiable function, a j -th section representative of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}^{(C)}$ (resp. $\theta_{G_R^{(n)}}^{(C)}$).
- $\phi_{G_{jL}^{(C)}}^{(n)}(z_{g_{jL}})$ (resp. $\phi_{G_{jR}^{(C)}}^{(n)}(z_{g_{jR}})$), submitted locally to contracting surjective morphism(s) of singularisation(s), can be:

- a) either a Morse function, i.e. a function having a non-degenerate singular point at zero where a local coordinate system (z_1, \dots, z_n) in $(\mathbb{C}^n, 0)$ exists for which:

$$\phi_{G_{jL}^{(C)}}^{(n)}(z_1, \dots, z_n) = \sum_{i=1}^n z_i^2 \quad (\text{resp. } \phi_{G_{jR}^{(C)}}^{(n)}(z_1^*, \dots, z_n^*) = \sum_{i=1}^n (z_i^*)^2);$$

- b) or a function having a degenerate singular point [Cam1], [Cam2] of type A_k , D_k , E_6 , E_7 or E_8 , as mentioned in section 2.19. In $(\mathbb{C}^n, 0) \approx (\mathbb{R}^{2n}, 0)$, a local real coordinate system (x_1, \dots, x_{2n}) can be found for which (case A_k):

$$\begin{aligned} \phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) &= x_{1_{L_j}}^{k+1} + \sum_{i=2}^{2n} x_{i_{L_j}}^2 \\ (\text{resp. } \phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(x_{1_{R_j}}, \dots, x_{2n_{R_j}}) &= x_{1_{R_j}}^{k+1} + \sum_{i=2}^{2n} x_{i_{R_j}}^2), \quad x_{i_R} = -x_{i_L}. \end{aligned}$$

A small deformation (for example, of versal type) allows to split up a compound singular point generally into simpler ones.

For instance, a small deformation of the function

$$\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) = x_{1_{L_j}}^{k+1} + \sum_{i=2}^{2n} x_{i_{L_j}}^2$$

allows to transform it into:

$$\tilde{\phi}_{G_{jL}^{(\mathbb{R})}}^{(2n)}(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) = \phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) - \varepsilon x_{1_{L_j}};$$

this is a function having k singular points [H-Z], generally simpler than the original one.

- All the sections of the left (resp. right) semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$), localized in some open ball, are assumed to be affected by the same kind of external perturbations, and, thus, by the same kind of singularities.
- Assume, on the other hand, that, instead of considering only the semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$), we have this semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$) covered by one or several semisheaves $\{\theta_{SOT(1)_L}^*, \theta_{SOT(2)_L}^*\}$ (resp. $\{\theta_{SOT(1)_R}^*, \theta_{SOT(2)_R}^*\}$) due to the versal deformations and spreading-out isomorphisms as described precedingly and especially in section 4.1.5. Then, we can state the following lemma.

5.1.4 Lemma

The monodromy on the sections of the semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$), covered partially by semisheaves generated by spreading-out isomorphisms, can not occur before the blow-up of these covering semisheaves.

Proof. The monodromy arises only in an expanding phase, i.e. in a phase whose distance between objects increases. So, as the semisheaves $\{\theta_{SOT(1)_L}^*, \theta_{SOT(2)_L}^*\}$ (resp. $\{\theta_{SOT(1)_R}^*, \theta_{SOT(2)_R}^*\}$), generated by blow-up of the versal deformations, cover only partially by patches the semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$), the expanding phase gives rise to:

- a) a blow-up of the covering semisheaves $\{\theta_{SOT(1)_L}^*, \theta_{SOT(2)_L}^*\}$ (resp. $\{\theta_{SOT(1)_R}^*, \theta_{SOT(2)_R}^*\}$) disconnecting them completely from $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$).
- b) contracting surjective morphisms of singularisations due to the strong perturbation of the expanding phase.
- c) monodromy groups on the sections $\phi_{G_{jL}^{(C)}}^{*(n)}$ (resp. $\phi_{G_{jR}^{(C)}}^{*(n)}$) of the semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{C})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{C})}$). ■

5.1.5 Monodromy for non degenerate singularities of corank $2n$

- Assume that each section $\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(x_{1_{L_j}}, \dots, x_{2n_{L_j}})$ (resp. $\phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(x_{1_{R_j}}, \dots, x_{2n_{R_j}})$) of the semisheaf $\theta_{G_L^{(n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(n)}}^{(\mathbb{R})}$) is a Morse function, i.e. a function affected by an

isolated non degenerate singularity on a domain U_{j_L} (resp. U_{j_R}) included into the conjugacy class representative $g_L^{(n)}[j, m_j]$ (resp. $g_R^{(n)}[j, m_j]$). Then, on the domain(s) $U_{j_L} \subset g_L^{(n)}[j, m_j]$ (resp. $U_{j_R} \subset g_R^{(n)}[j, m_j]$), $\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L})$ (resp. $\phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R})$) is described locally by:

$$\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L}) = \sum_{i=1}^{2n} x_{i_{L_j}}^2 \quad (\text{resp.} \quad \phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R}) = \sum_{i=1}^{2n} x_{i_{R_j}}^2).$$

- The critical level set of $\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L})$ (resp. $\phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R})$) is the singular fibre $F_{0_{j_L}}^{(2n-1)}$ (resp. $F_{0_{j_R}}^{(2n-1)}$) given by

$$\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}(U_{j_L}) = \sum_{i=1}^{2n} x_{i_{L_j}}^2 = 0 \quad (\text{resp.} \quad \phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}(U_{j_R}) = \sum_{i=1}^{2n} x_{i_{R_j}}^2 = 0).$$

- The non-singular fibres $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) of $\phi_{G_{j_L}^{(\mathbb{R})}}^{(2n)}$ (resp. $\phi_{G_{j_R}^{(\mathbb{R})}}^{(2n)}$) are diffeomorphic to the space $TS_{L_j}^{2n-1}$ (resp. $TS_{R_j}^{2n-1}$) of the tangent bundle to a sphere $S_{L_j}^{2n-1}$ (resp. $S_{R_j}^{2n-1}$) of real dimension $(2n-1)$ and radius $r_{L_j} = 1$ (resp. $r_{R_j} = 1$).

The non-singular fibres $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) are thus diffeomorphic to:

$$TS_{L_j}^{2n-1} = \{(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) \mid x_{1_{L_j}}^2 + \dots + x_{2n_{L_j}}^2 = \lambda_{L_j}\} \\ (\text{resp.} \quad TS_{R_j}^{2n-1} = \{(x_{1_{R_j}}, \dots, x_{2n_{R_j}}) \mid x_{1_{R_j}}^2 + \dots + x_{2n_{R_j}}^2 = \lambda_{R_j}\})$$

while the sphere

$$S_{L_j}^{2n-1} = \{(x_{1_{L_j}}, \dots, x_{2n_{L_j}}) \mid x_{1_{L_j}}^2 + \dots + x_{2n_{L_j}}^2 = 1\} \\ (\text{resp.} \quad S_{R_j}^{2n-1} = \{(x_{1_{R_j}}, \dots, x_{2n_{R_j}}) \mid x_{1_{R_j}}^2 + \dots + x_{2n_{R_j}}^2 = 1\})$$

is diffeomorphic to the vanishing cycle $\Delta_{L_j}^{(2n-1)} \subset F_{\lambda_{L_j}}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)} \subset F_{\lambda_{R_j}}^{(2n-1)}$).

- As the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) is diffeomorphic to the unit sphere $S_{L_j}^{2n-1}$ (resp. $S_{R_j}^{2n-1}$), it must correspond to a function $\phi_{P_{L_j}}^{(2n-1)}(x_{p_{j_L}})$ (resp. $\phi_{P_{R_j}}^{(2n-1)}(x_{p_{j_R}})$) on the left (resp. right) real conjugacy class representative $P_{v_1^{j,m_j}}^{(2n-1)}(F_{v_1^{j,m_j}}^+)$ (resp. $P_{\bar{v}_1^{j,m_j}}^{(2n-1)}(F_{\bar{v}_1^{j,m_j}}^+)$) of the left (resp. right) linear parabolic subgroup $P^{(2n-1)}(F_{v_1^+})$ (resp. $P^{(2n-1)}(F_{\bar{v}_1^+})$). Indeed, according to [Pie1], the bilinear parabolic affine subsemigroup $P^{(2n-1)}(F_{\bar{v}_1^+} \times F_{v_1^+})$ can be considered as the unitary irreducible representation (space) of the algebraic bilinear semigroup $\text{GL}_{2n-1}(F_{\bar{v}}^+ \times F_v^+)$.

- Let $\gamma_{jL} : [0, 1] \rightarrow \Delta_{L_j}^{(2n-1)}$ (resp. $\gamma_{jR} : [0, 1] \rightarrow \Delta_{R_j}^{(2n-1)}$) be a closed loop on the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$).

5.1.6 Proposition

Let $(\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(U_{jL}), F_{0_{jL}}^{(2n-1)}, F_{\lambda_{jL}}^{(2n-1)}, \Delta_{L_j}^{(2n-1)}, \gamma_{jL})$ (resp. $(\phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(U_{jR}), F_{0_{jR}}^{(2n-1)}, F_{\lambda_{jR}}^{(2n-1)}, \Delta_{R_j}^{(2n-1)}, \gamma_{jR})$) be the 5-th tuple introduced in section 5.1.5.

Then, the mapping

$$h_{\gamma_{jL}} : F_{\lambda_{jL}}^{(2n-1)} \longrightarrow F_{\lambda_{jL}}^{(2n-1)} \quad (\text{resp. } h_{\gamma_{jR}} : F_{\lambda_{jR}}^{(2n-1)} \longrightarrow F_{\lambda_{jR}}^{(2n-1)})$$

of the non-singular fibre $F_{\lambda_{jL}}^{(2n-1)}$ (resp. $F_{\lambda_{jR}}^{(2n-1)}$) into itself is the monodromy of the closed loop γ_{jL} (resp. γ_{jR}) realized by the conjugation action of the j -th conjugacy class representative of the restricted linear algebraic semigroup $G^{(2n-1)}(F_{v_j}^{+\text{res}})$ (resp. $G^{(2n-1)}(F_{\bar{v}_j}^{+\text{res}})$) on the j -th conjugacy class representative of the linear parabolic subsemigroup $P^{(2n-1)}(F_{v_j^1}^+)$ (resp. $P^{(2n-1)}(F_{\bar{v}_j^1}^+)$) where $F_{v_j}^{+\text{res}}$ (resp. $F_{\bar{v}_j}^{+\text{res}}$) is the j -th real completion restricted to the domain U_{jL} (resp. U_{jR}).

Proof.

- 1) The monodromy $h_{\gamma_{jL}}$ (resp. $h_{\gamma_{jR}}$) is associated with an injective mapping

$$\begin{aligned} I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}} : \Delta_{L_j}^{(2n-1)} &\longrightarrow F_{\lambda_{jL}}^{(2n-1)} \\ (\text{resp. } I_{\Delta_{R_j} \rightarrow F_{\lambda_{jR}}} : \Delta_{R_j}^{(2n-1)} &\longrightarrow F_{\lambda_{jR}}^{(2n-1)}) \end{aligned}$$

inflating the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) into the non-singular fibre $F_{\lambda_{jL}}^{(2n-1)}$ (resp. $F_{\lambda_{jR}}^{(2n-1)}$). This injective mapping $I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}}$ (resp. $I_{\Delta_{R_j} \rightarrow F_{\lambda_{jR}}}$) is in one-to-one correspondence with the injective mapping

$$\begin{aligned} I_{S_{L_j}^{2n-1} \rightarrow TS_{L_j}^{2n-1}} : S_{L_j}^{2n-1} &\longrightarrow TS_{L_j}^{2n-1} \\ (\text{resp. } I_{S_{R_j}^{2n-1} \rightarrow TS_{R_j}^{2n-1}} : S_{R_j}^{2n-1} &\longrightarrow TS_{R_j}^{2n-1}), \end{aligned}$$

i.e. the inverse of the projective mapping of the tangent bundle introduced in section 5.1.5.

$I_{S_{L_j}^{2n-1} \rightarrow TS_{L_j}^{2n-1}}$ (resp. $I_{S_{R_j}^{2n-1} \rightarrow TS_{R_j}^{2n-1}}$) then inflates the sphere $S_{L_j}^{2n-1}$ (resp. $S_{R_j}^{2n-1}$), diffeomorphic to the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$), into $TS_{L_j}^{2n-1}$ (resp.

$TS_{R_j}^{2n-1}$), diffeomorphic to the non-singular fibre $F_{\lambda_{jL}}^{(2n-1)}$ (resp. $F_{\lambda_{jR}}^{(2n-1)}$), in such a way that the following diagram:

$$\begin{array}{ccc} \Delta_{L_j}^{(2n-1)} & \xrightarrow{I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}}} & F_{\lambda_{jL}}^{(2n-1)} \\ \downarrow \wr & & \downarrow \wr \\ S_{L_j}^{2n-1} & \xrightarrow{I_{S_{L_j}^{2n-1} \rightarrow TS_{L_j}^{2n-1}}} & TS_{L_j}^{2n-1} \end{array}$$

be commutative.

- 2) This injective mapping $I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}}$ (resp. $I_{\Delta_{R_j} \rightarrow F_{\lambda_{jR}}}$) corresponds to the conjugation action of the j -th conjugacy class representative $G^{(2n-1)}(F_{v_j}^{+(\text{res})})$ (resp. $G^{(2n-1)}(F_{\bar{v}_j}^{+(\text{res})})$) of $G^{(2n-1)}(F_v^{+(\text{res})})$ (resp. $G^{(2n-1)}(F_{\bar{v}}^{+(\text{res})})$) on the j -th conjugacy class representative $P^{(2n-1)}(F_{v_j}^+)$ (resp. $P^{(2n-1)}(F_{\bar{v}_j}^+)$) of the linear parabolic subsemigroup, as developed in [Pie1], since the linear parabolic subsemigroup $P^{(2n-1)}(F_{v_1}^{+(\text{res})})$ (resp. $P^{(2n-1)}(F_{\bar{v}_1}^{+(\text{res})})$) can be considered as the unitary irreducible representation space of $\text{GL}_{2n-1}(F_v^{+(\text{res})})$ (resp. $\text{GL}_{2n-1}(F_{\bar{v}}^{+(\text{res})})$).
- 3) The set of non-singular fibres $F_{\lambda_{jL}}^{(2n-1)}(t)$ (resp. $F_{\lambda_{jR}}^{(2n-1)}(t)$), $t \in [0, 1]$ of γ_{jL} (resp. γ_{jR}), generates a sheaf $\mathcal{F}_{F_{\lambda_{jL}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{\lambda_{jR}}^{(2n-1)}}$) on the etale sites above U_{jL} (resp. U_{jR}) [Del1], [Del2]. ■

5.1.7 Proposition

The injective mapping

$$\begin{aligned} I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}} : \quad \Delta_{L_j}^{(2n-1)} &\longrightarrow F_{\lambda_{jL}}^{(2n-1)} \\ (\text{resp. } I_{\Delta_{R_j} \rightarrow F_{\lambda_{jR}}} : \quad \Delta_{R_j}^{(2n-1)} &\longrightarrow F_{\lambda_{jR}}^{(2n-1)}), \end{aligned}$$

being the inverse of the projective mapping of the tangent bundle $TB_{jL}(\Delta_{L_j}^{(2n-1)}, F_{\lambda_{jL}}^{(2n-1)}, I_{\Delta_{L_j} \rightarrow F_{\lambda_{jL}}}^{-1})$ (resp. $TB_{jR}(\Delta_{R_j}^{(2n-1)}, F_{\lambda_{jR}}^{(2n-1)}, I_{\Delta_{R_j} \rightarrow F_{\lambda_{jR}}}^{-1})$), is such that:

- 1) the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) is characterized by a rank $r_{\Delta_j^{(2n-1)}} = N^{2n-1}$.
- 2) the non-singular fibre $F_{\lambda_{jL}}^{(2n-1)}$ (resp. $F_{\lambda_{jR}}^{(2n-1)}$) is characterized by a rank $r_{F_{\lambda_j}^{(2n-1)}} \leq (j \cdot N)^{2n-1}$.

- 3) the fibre $F_{I_{\Delta_{L_j} \rightarrow F_{\lambda_{j_L}}}}$ (resp. $F_{I_{\Delta_{R_j} \rightarrow F_{\lambda_{j_R}}}}$) of the tangent bundle TB_{j_L} (resp. TB_{j_R}) has a rank $r_{F_{I_{\Delta_{L_j} \rightarrow F_{\lambda_{j_L}}}}}$ verifying:

$$r_{F_{I_{\Delta_{L_j} \rightarrow F_{\lambda_{j_L}}}}} \leq (j \cdot N)^{2n-1}.$$

Proof.

- 1) According to section 5.1.5, the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) is a function on the j -th conjugacy class representative of the parabolic subgroup $P^{(2n-1)}(F_{v^1}^+)$ (resp. $P^{(2n-1)}(F_{\bar{v}^1}^+)$). So we have that its rank is given by [Pie1]:

$$r_{\Delta_{L_j}^{(2n-1)}} = N^{2n-1}.$$

- 2) As $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) is a non-singular fibre above the j -th conjugacy class representative of the linear algebraic semigroup $G^{(n)}(F_v^+)$ (resp. $G^{(n)}(F_{\bar{v}}^+)$) characterized by a rank:

$$r_{g^{(n)}[j, m_j]} = (j \cdot N)^{2n}$$

and as $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) results from an inflation mapping from the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) in such a way that the inflation of $F_{\lambda_{j_L}}^{(2n-1)}$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}$) from $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) is proportional to the conjugacy action of the j -th conjugacy class representative $g_L^{(n)}[j, m_j] \in G^{(n)}(F_v^+)$ (resp. $g_R^{(n)}[j, m_j] \in G^{(n)}(F_{\bar{v}}^+)$) with respect to the j -th conjugacy class representative of the parabolic subgroup $P^{(2n)}(F_{v^1}^+)$ (resp. $P^{(2n)}(F_{\bar{v}^1}^+)$), we have that:

$$r_{F_{\lambda_j}^{(2n-1)}} \leq (j \cdot N)^{2n-1}$$

since $(j \cdot N)^{2n}$ is the rank of $g_L^{(n)}[j, m_j]$ (resp. $g_R^{(n)}[j, m_j]$). ■

5.1.8 Definition: Monodromy operator [Ber1], [Ber2], [Chm]

If we have that

$$\begin{aligned} h_{\gamma_{j_L}^*} : H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}}; \mathbb{Z}) &\longrightarrow H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}}; \mathbb{Z}) \\ (\text{resp. } h_{\gamma_{j_R}^*} : H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}}; \mathbb{Z}) &\longrightarrow H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}}; \mathbb{Z})), \end{aligned}$$

the action $h_{\gamma_{j_L}^*}$ (resp. $h_{\gamma_{j_R}^*}$) of $h_{\gamma_{j_L}}$ (resp. $h_{\gamma_{j_R}}$) in the homology group $H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}}; \mathbb{Z})$ (resp. $H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}}; \mathbb{Z})$) of the sheaf $\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}}$) of non-singular fibres

$F_{\lambda_{j_L}}^{(2n-1)}(t)$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}(t)$), is the monodromy operator of the closed loop γ_{j_L} (resp. γ_{j_R}) [H-Z].

Indeed, the homology group $H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}}; \mathbb{Z})$ (resp. $H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}}; \mathbb{Z})$) is generated by the homology class of the vanishing cycle $\Delta_{L_j}^{(2n-1)}$ (resp. $\Delta_{R_j}^{(2n-1)}$) [H-Z].

5.1.9 Definition: The surjective mapping

The surjective mapping

$$\begin{aligned} r_{F_{\lambda_{j_L}} \rightarrow F_{0_{j_L}}} : F_{\lambda_{j_L}}^{(2n-1)}(t) &\longrightarrow F_{0_{j_L}}^{(2n-1)}, & 0 \leq t \leq 1 \\ \text{(resp. } r_{F_{\lambda_{j_R}} \rightarrow F_{0_{j_R}}} : F_{\lambda_{j_R}}^{(2n-1)}(t) &\longrightarrow F_{0_{j_R}}^{(2n-1)}) \end{aligned}$$

of the non-singular fibre(s) $F_{\lambda_{j_L}}^{(2n-1)}(t)$ (resp. $F_{\lambda_{j_R}}^{(2n-1)}(t)$) into the singular fibre $F_{0_{j_L}}^{(2n-1)}$ (resp. $F_{0_{j_R}}^{(2n-1)}$) is the retraction of the monodromy.

5.1.10 Monodromy for degenerate singularities

If a degenerate singularity (for example of type A_k (see section 5.1.3)) decomposes by deformation into k elementary non degenerate singular points, the single monodromy envisaged in the case of a unique non degenerate singularity, becomes a monodromy group where the loop γ_{j_L} (resp. γ_{j_R}) runs through the fundamental group $\Pi_1(V_{j_L} - \{\omega_{i_L}\}, x_{0_L})$ (resp. $\Pi_1(V_{j_R} - \{\omega_{i_R}\}, x_{0_R})$) of the complementary of the set of critical values ω_{i_L} (resp. ω_{i_R}), $1 \leq i \leq k$, where:

- V_{j_L} (resp. V_{j_R}) is a compact domain in \mathbb{C} included into U_{j_L} (resp. U_{j_R});
- $\gamma_{j_L}(0) = \gamma_{j_L}(1) = x_{0_L}$ (resp. $\gamma_{j_R}(0) = \gamma_{j_R}(1) = x_{0_R}$).

The complementary of the set of critical values in V_{j_L} (resp. V_{j_R}) is a loop beginning and ending at x_{0_L} (resp. x_{0_R}) and passing round the critical values ω_{i_L} (resp. ω_{i_R}).

The domain V_{j_L} (resp. V_{j_R}) without the k critical values $\{\omega_{i_L} \mid i = 1, \dots, k\}$ (resp. $\{\omega_{i_R} \mid i = 1, \dots, k\}$) of $\phi_{G_{j_L}^{(2n)}}^{(2n)}(x_{i_L})$ (resp. $\phi_{G_{j_R}^{(2n)}}^{(2n)}(x_{i_R})$) is homotopically equivalent to a bunch of k circles. So, the fundamental group $\Pi_1(V_{j_L} - \{\omega_{i_L}\}, x_{0_L})$ (resp. $\Pi_1(V_{j_R} - \{\omega_{i_R}\}, x_{0_R})$) is a free group at k generators.

If $\{v_{i_L} \mid i = 1, \dots, k\}$ (resp. $\{v_{i_R} \mid i = 1, \dots, k\}$) is a set of paths defining a set of vanishing cycles $\Delta_{i_{L_j}}^{(2n-1)} \in H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_L}}^{(2n-1)}})$ (resp. $\Delta_{i_{R_j}}^{(2n-1)} \in H_{2n-1}(\mathcal{F}_{F_{\lambda_{j_R}}^{(2n-1)}})$), $1 \leq i \leq k$, then the fundamental group $\Pi_1(V_{j_L} - \{\omega_{i_L}\}, x_{0_L})$ (resp. $\Pi_1(V_{j_R} - \{\omega_{i_R}\}, x_{0_R})$) is generated by the simple loops $\gamma_{1_{j_L}}, \dots, \gamma_{k_{j_L}}$ (resp. $\gamma_{1_{j_R}}, \dots, \gamma_{k_{j_R}}$) associated with the paths v_{1_L}, \dots, v_{k_L} (resp. v_{1_R}, \dots, v_{k_R}).

The monodromy group of $\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(U_{jL})$ (resp. $\phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(U_{jR})$) is the image of the homomorphism of the fundamental group $\Pi_1(V_{jL} - \{\omega_{i_L}\}, x_{0_L})$ (resp. $\Pi_1(V_{jR} - \{\omega_{i_R}\}, x_{0_R})$) into the group $\text{Aut}(H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}))$ (resp. $\text{Aut}(H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}))$) of automorphisms of $H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}})$ (resp. $H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}})$) which associate with the loop $\gamma_{i_{jL}}$ (resp. $\gamma_{i_{jR}}$) the monodromy operator:

$$\begin{aligned} h_{\gamma_{i_{jL}}}^* &: H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}; \mathbb{Z}) \longrightarrow H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}; \mathbb{Z}) \\ (\text{resp. } h_{\gamma_{i_{jR}}}^* &: H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}; \mathbb{Z}) \longrightarrow H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}; \mathbb{Z})), \end{aligned}$$

where $\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}$) is the i -th sheaf of non-singular fibres $F_{i\lambda_{jL}}^{(2n-1)}(t)$ (resp. $F_{i\lambda_{jR}}^{(2n-1)}(t)$).

As developed in [Pie1] and in proposition 5.1.6,

$$\begin{aligned} \text{Aut}(H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}})) &= \text{Aut}(\Delta_{iL_j}^{(2n-1)}) \simeq \text{Aut}(P^{(2n-1)}(F_{v_j^+}^+)) \\ (\text{resp. } \text{Aut}(H_{2n-1}(\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}})) &= \text{Aut}(\Delta_{iR_j}^{(2n-1)}) \simeq \text{Aut}(P^{(2n-1)}(F_{\bar{v}_j^+}^+)) \end{aligned}$$

where $\text{Aut}(P^{(2n-1)}(F_{v_j^+}^+))$ (resp. $\text{Aut}(P^{(2n-1)}(F_{\bar{v}_j^+}^+))$) is the group of Galois automorphisms of the linear parabolic subsemigroup $P^{(2n-1)}(F_{v_j^+}^+)$ (resp. $P^{(2n-1)}(F_{\bar{v}_j^+}^+)$) on the j -th irreducible completion $F_{v_j^+}^+$ (resp. $F_{\bar{v}_j^+}^+$) of rank N .

5.1.11 Monodromy for a set of non degenerate singularities

The monodromy, envisaged in section 5.1.9 for a degenerate singularity splitting up into elementary non degenerate singularities, is also valid for a set of critical points ω_{i_L} (resp. ω_{i_R}), $1 \leq i \leq k$, of $\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(\omega_{i_L})$ (resp. $\phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(\omega_{i_R})$) which are not degenerated and such that their critical values ω_{i_L} (resp. ω_{i_R}) are distinct.

Then, we can state the following proposition.

5.1.12 Proposition

Assume that every section $\phi_{G_{jL}^{(\mathbb{R})}}^{(2n)}(U_{jL})$ (resp. $\phi_{G_{jR}^{(\mathbb{R})}}^{(2n)}(U_{jR})$) of the semisheaf $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$), $1 \leq j \leq r \leq \infty$, is endowed with a set of k non degenerate singularities ω_{i_L} (resp. ω_{i_R}), $1 \leq i \leq k$, on the domain U_{jL} (resp. U_{jR}).

Let $(\phi_{G_{jL}^{(2n)}}^{(2n)}(U_{jL}), F_{i_{0jL}}^{(2n-1)}, \mathcal{F}_{F_{i_{\lambda jL}}^{(2n-1)}}, \Delta_{i_{Lj}}^{(2n-1)}, \gamma_{i_{jL}})$ (resp. $(\phi_{G_{jR}^{(2n)}}^{(2n)}(U_{jR}), F_{i_{0jR}}^{(2n-1)}, \mathcal{F}_{F_{i_{\lambda jR}}^{(2n-1)}}, \Delta_{i_{Rj}}^{(2n-1)}, \gamma_{i_{jR}})$) be the i -th 5-tuple associated with the i -th singularity on the j -th section of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$), as introduced in proposition 5.1.6.

Then, the i mappings:

$$\begin{aligned} h_{\gamma_{i_{jL}}} &: F_{i_{\lambda jL}}^{(2n-1)} \longrightarrow F_{i_{\lambda jL}}^{(2n-1)}, \quad 1 \leq i \leq k, \\ (\text{resp. } h_{\gamma_{i_{jR}}} &: F_{i_{\lambda jR}}^{(2n-1)} \longrightarrow F_{i_{\lambda jR}}^{(2n-1)}) \end{aligned}$$

of the non-singular fibres $F_{i_{\lambda jL}}^{(2n-1)}$ (resp. $F_{i_{\lambda jR}}^{(2n-1)}$) into themselves, associated with the “ i ” non degenerate singularities ω_{i_L} (resp. ω_{i_R}):

- 1) are the monodromies of the closed loops $\gamma_{i_{jL}}$ (resp. $\gamma_{i_{jR}}$);
- 2) generate the sheaves $\mathcal{F}_{F_{i_{\lambda jL}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{i_{\lambda jR}}^{(2n-1)}}$) of non singular fibres $F_{i_{\lambda jL}}^{(2n-1)}(t)$ (resp. $F_{i_{\lambda jR}}^{(2n-1)}(t)$) on the etale sites U_{jL} (resp. U_{jR});
- 3) are associated with the injective mappings:

$$\begin{aligned} I_{\Delta_{i_{Lj}} \rightarrow F_{i_{\lambda jL}}} &: \Delta_{i_{Lj}}^{(2n-1)} \longrightarrow F_{i_{\lambda jL}}^{(2n-1)}(t) \\ (\text{resp. } I_{\Delta_{i_{Rj}} \rightarrow F_{i_{\lambda jR}}} &: \Delta_{i_{Rj}}^{(2n-1)} \longrightarrow F_{i_{\lambda jR}}^{(2n-1)}(t)) \end{aligned}$$

inflating the vanishing cycles $\Delta_{i_{Lj}}^{(2n-1)}$ (resp. $\Delta_{i_{Rj}}^{(2n-1)}$) into the non-singular fibres $F_{i_{\lambda jL}}^{(2n-1)}(t)$ (resp. $F_{i_{\lambda jR}}^{(2n-1)}(t)$) and being in one-to-one correspondence with the corresponding injective mappings:

$$\begin{aligned} I_{S_{i_{Lj}}^{2n-1} \rightarrow TS_{i_{Lj}}^{2n-1}} &: S_{i_{Lj}}^{2n-1} \longrightarrow TS_{i_{Lj}}^{2n-1} \\ (\text{resp. } I_{S_{i_{Rj}}^{2n-1} \rightarrow TS_{i_{Rj}}^{2n-1}} &: S_{i_{Rj}}^{2n-1} \longrightarrow TS_{i_{Rj}}^{2n-1}); \end{aligned}$$

- 4) result from the monodromy operators:

$$\begin{aligned} h_{\gamma_{i_{jL}}}^* &: H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jL}}^{(2n-1)}}) \longrightarrow H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jL}}^{(2n-1)}}) \\ (\text{resp. } h_{\gamma_{i_{jR}}}^* &: H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jR}}^{(2n-1)}}) \longrightarrow H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jR}}^{(2n-1)}})) \end{aligned}$$

on the vanishing cycles $H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jL}}^{(2n-1)}}) \equiv \Delta_{i_{Lj}}^{(2n-1)}$ (resp. $H_{2n-1}(\mathcal{F}_{F_{i_{\lambda jR}}^{(2n-1)}}) \equiv \Delta_{i_{Rj}}^{(2n-1)}$).

5.1.13 Monodromy sheaves above $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$)

Let $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ be the bisemisheaf on the bilinear affine semigroup $G^{(2n)}(F_v^+ \times F_v^+)$.

If every section of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$) is endowed with k isolated non degenerate singularities, then the set of bisheaves $\{\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}} \otimes \mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}\}_{i=1}^k$ of non singular bifibres $F_{i\lambda_{jR}}^{(2n-1)}(t) \otimes F_{i\lambda_{jL}}^{(2n-1)}(t)$ are generated by monodromy above every bisection $\phi_{G_{jR}^{(2n)}}^{(2n)}(U_{jR}) \otimes \phi_{G_{jL}^{(2n)}}^{(2n)}(U_{jL})$ of $(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})})$.

Let $\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}$) be the sheaf of non-singular fibres of the monodromy of the closed loop $\gamma_{i\lambda_{jL}}$ (resp. $\gamma_{i\lambda_{jR}}$) associated with the i -th singularity on the j -th section of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$).

Consider that there are b_i , $b_i \in \mathbb{N}$, non-singular fibres in the sheaf $\mathcal{F}_{F_{i\lambda_{jL}}^{(2n-1)}}$ (resp. $\mathcal{F}_{F_{i\lambda_{jR}}^{(2n-1)}}$).

As it was assumed that all the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$) are endowed with the same kind of k isolated non degenerate singularities, b_i sheaves $\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$), $1 \leq \beta_i \leq b_i$, whose sections are the non-singular fibres above the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$), can be envisaged as generated by monodromy from the i -th singularities on all the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$).

So, above the i -th singularity on all the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$), a set $\{\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{\beta_i=1}^{b_i}$ (resp. $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)\}_{\beta_i=1}^{b_i}$) of b_i monodromy sheaves are generated and, above the set of i singularities on all the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$), a set of $i \times b_i$ monodromy sheaves can be constructed above $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$).

Let $\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$) be the β_i -th monodromy sheaf, generated from the i -th singularity on all the sections of $\theta_{G_L^{(2n)}}^{(\mathbb{R})}$ (resp. $\theta_{G_R^{(2n)}}^{(\mathbb{R})}$).

The sections of $\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$) are in one-to-one correspondence with the conjugacy class representatives of $G_L^{(2n)}(F_v^+)$ (resp. $G_R^{(2n)}(F_v^+)$): they are thus labelled by the pairs of integers (j, m_j) , $1 \leq j \leq r \leq \infty$, and their ranks verify:

$$r_{F_{i\lambda_j}^{(2n-1)}} \leq (j \cdot N)^{2n-1},$$

according to proposition 5.1.7, $F_{i\lambda_{jL}}^{(2n-1)}(\beta_i) \in \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $F_{i\lambda_{jR}}^{(2n-1)}(\beta_i) \in \mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$) being the j -th section.

5.1.14 Proposition

Let $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \times \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ be the bisemisheaf on $G^{(2n)}(F_v^+ \times F_v^+)$ such that its linear sections are endowed with k isolated non-degenerate singularities of the same type.

Let $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{i,\beta_i}$, $1 \leq i \leq k$, $1 \leq \beta_i \leq b_i$, be the set of $k \times b_i$ monodromy bi(semi)sheaves above $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ according to section 5.1.13.

Then, it results that:

- 1) a global holomorphic representation $\text{Irr hol}^{(2n)}(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})})$ of the bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ is given by the morphism::

$$\text{Irr hol}_{\theta_{G_R \times L}^{(2n)}}^{(2n)} : \theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z)$$

where $f_{\bar{v}}(z^*) \otimes f_v(z)$ is the holomorphic bifunction getting by gluing together and adding the bisections of the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.

- 2) $k \times b_i$ global holomorphic representations

$$\text{Irr hol}_{\mathcal{F}_{F_{i\lambda_R \times L}^{(2n-1)}}}^{(2n-1)} : \mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i) \longrightarrow f_{\bar{v}_{\text{mon}}}(z_{\beta_i}^*) \otimes f_{v_{\text{mon}}}(z_{\beta_i}),$$

$$\forall 1 \leq i \leq k, 1 \leq \beta_i \leq b_i$$

of the monodromy bisemisheaves $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ can be stated

where $f_{\bar{v}_{\text{mon}}}(z_{\beta_i}^*) \otimes f_{v_{\text{mon}}}(z_{\beta_i})$ is the (i, β_i) -th holomorphic bifunction obtained by gluing and adding the sections of $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$.

Proof.

- 1) If the bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ is desingularized, then a $2n$ -dimensional irreducible global holomorphic representation can be envisaged for it, as it was done in section 4.2.3, in the sense that a multiple power series development $f_{\bar{v}}(z^*) \otimes f_v(z)$ can be associated to it where

$$f_v(z) = \sum_{j, m_j} c_{j, m_j} (z_1 - z_{01})^j \cdots (z_{2n} - z_{02n})^j$$

$$(\text{resp. } f_{\bar{v}}(z^*) = \sum_{j, m_j} c_{j, m_j}^* (z_1^* - z_{01}^*)^j \cdots (z_{2n}^* - z_{02n}^*)^j)$$

with:

- $z_1, z_{0n}, \dots, z_{2n}, z_{02n}$ are complex functions of one real variable;

- c_{j,m_j} is in one-to-one correspondence with the product of the square roots of the eigenvalues of the (j, m_j) -th coset representative $U_{j,m_{j_R}} \times U_{j,m_{j_L}}$ of the product of Hecke operators;
- the sum \sum_{j,m_j} runs over the conjugacy class representatives of $G_L^{(2n)}$ (resp. $G_R^{(2n)}$) which are glued together.

2) Assume that the sections $F_{i_{\lambda_{j_L}}^{(2n-1)}}(\beta_i)$ (resp. $F_{i_{\lambda_{j_R}}^{(2n-1)}}(\beta_i)$) of the monodromy sheaf $\mathcal{F}_{F_{i_{\lambda_L}}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i_{\lambda_R}}^{(2n-1)}}(\beta_i)$) have ranks given by $r_{F_{i_{\lambda_j}}^{(2n-1)}} = (j \cdot N)^{2n-1}$ according to section 5.1.13.

If these sections, which are complex-valued differentiable functions, are glued together, a holomorphic function (resp. cofunction) given by the multiple power series development:

$$f_{v_{\text{mon}}}(z_{\beta_i}) = \sum_{j,m_j} c_{j\beta_i,m_{j\beta_i}} (z_{1\beta_i} - z_{01\beta_i})^j \cdots (z_{2n-1\beta_i} - z_{0(2n-1)\beta_i})^j$$

$$(\text{resp. } f_{\overline{v}_{\text{mon}}}(z_{\beta_i}^*) = \sum_{j,m_j} c_{j\beta_i,m_{j\beta_i}}^* (z_{1\beta_i}^* - z_{01\beta_i}^*)^j \cdots (z_{2n-1\beta_i}^* - z_{0(2n-1)\beta_i}^*)^j)$$

can be associated with them (see, for instance, section 4.2.4). And, a global holomorphic representation $\text{Irr hol}_{\mathcal{F}_{F_{i_{\lambda_{R \times L}}^{(2n-1)}}}}^{(2n-1)}$ of the monodromy bisemisheaf $\mathcal{F}_{F_{i_{\lambda_R}}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i_{\lambda_L}}^{(2n-1)}}(\beta_i)$, as given in this proposition, can be envisaged. ■

5.1.15 Monodromy n -dimensional representations of global Weil groups

As the monodromy bisemisheaves $\mathcal{F}_{F_{i_{\lambda_R}}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i_{\lambda_L}}^{(2n-1)}}(\beta_i)$ are of algebraic type and are defined above the algebraic bilinear semigroup $G^{(n)}(F_v^+ \times F_v^+)$, $(2n-1)$ -dimensional irreducible real representations $\text{Irr Rep}_{W_{F_{R \times L}}^{(2n-1)}}(W_{F_{R_{\text{mon}}}}^{ab}(\beta_i) \times W_{F_{L_{\text{mon}}}}^{ab}(\beta_i))$ of the bilinear global Weil groups $(W_{F_{R_{\text{mon}}}}^{ab}(\beta_i) \times W_{F_{L_{\text{mon}}}}^{ab}(\beta_i))$ can be introduced for them, similarly as it was done in section 4.2.5.

And, as it was developed in section 4.1.1, the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ on the algebraic bilinear semigroup $G^{(2n)}(F_v^+ \times F_v^+)$ constitutes a $2n$ -dimensional irreducible representation $\text{Irr Rep}_{W_{F_{R \times L}}^{(2n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab})$ of the product, right by left, $W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}$ of global Weil groups.

5.1.16 Proposition

Let $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{i,\beta_i}$, $1 \leq i \leq k$, $1 \leq \beta_i \leq b_i$, be the set of $k \times \beta_i$ monodromy bisemisheaves above the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.

Then, the following global holomorphic correspondences are:

$$a) \quad \text{Irr Rep}_{W_{F_R^+ \times L}^{(2n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) \longrightarrow \text{Irr hol}^{(2n)}(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})})$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z)$$

for the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$;

$$b) \quad \text{Irr Rep}_{W_{F_R^+ \times L}^{(2n-1)}}(W_{F_{R\text{mon}}^+}^{ab}(\beta_i) \times W_{F_{L\text{mon}}^+}^{ab}(\beta_i)) \rightarrow \text{Irr hol}^{(2n-1)}(\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i))$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i) \rightarrow f_{\bar{v}_{\text{mon}}}(z_{\beta_i}^*) \otimes f_{v_{\text{mon}}}(z_{\beta_i})$$

$\forall i, \beta_i$,

for the monodromy bisemisheaves.

5.1.17 Toroidal compactification of $G^{(2n)}(F_v^+ \times F_v^+)$

As in section 4.2.7, a toroidal compactification of the bilinear algebraic semigroup $G^{(2n)}(F_v^+ \times F_v^+)$ can be envisaged in such a way that its linear conjugacy class representatives $g_L^{(2n)}[j, m_j]$ (resp. $g_R^{(2n)}[j, m_j]$) be transformed into $2n$ -dimensional real semitori $g_{T_L}^{(2n)}[j, m_j]$ (resp. $g_{T_R}^{(2n)}[j, m_j]$).

Let

$$\tau^{\text{tor}}[j, m_j] : \quad g_L^{(2n)}[j, m_j] \longrightarrow g_{T_L}^{(2n)}[j, m_j]$$

$$c_{j, m_j} z_j \longrightarrow \lambda^{\frac{1}{2}}(2n, j, m_j) e^{2\pi i j x}, \quad x \in \mathbb{R}^{2n},$$

$$(\text{resp. } \tau^{\text{tor}}[j, m_j] : \quad g_R^{(2n)}[j, m_j] \longrightarrow g_{T_R}^{(2n)}[j, m_j]$$

$$c_{j, m_j}^* z_j^* \longrightarrow \lambda^{\frac{1}{2}}(2n, j, m_j) e^{-2\pi i j x})$$

be the toroidal deformation of $g_L^{(2n)}[j, m_j]$ (resp. $g_R^{(2n)}[j, m_j]$) transforming $G^{(2n)}(F_v^+ \times F_v^+)$ into its toroidal equivalent $G^{(2n)}(F_v^{+,T} \times F_v^{+,T})$.

5.1.18 Proposition

Let

$$\text{Irr hol}_{\theta_{G_{R \times L}}^{(\mathbb{R})}}^{(2n-1)} : \theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z)$$

be the global holomorphic representation of the bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ on $G^{(2n)}(F_{\bar{v}}^+ \times F_v^+)$.

Then, the toroidal compactification $\tau^{\text{tor}}(\text{Irr hol}_{\theta_{G_{R \times L}}^{(\mathbb{R})}}^{(2n)})$ of the global holomorphic representation of the bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ generates the corresponding elliptic representation according to:

$$\begin{array}{ccc} \text{Irr hol}_{\theta_{G_{R \times L}}^{(\mathbb{R})}}^{(2n)} : & \theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} & \longrightarrow f_{\bar{v}}(z^*) \otimes f_v(z) \\ \downarrow \tau^{\text{tor}}(\text{Irr hol}_{\theta_{G_{R \times L}}^{(\mathbb{R})}}^{(2n)}) & \downarrow & \downarrow \\ \text{Irr ELLIP}_{\theta_{G_{R \times L}}^{(\mathbb{R})}}^{(2n)} : & \theta_{G_{T_R}^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_{T_L}^{(2n)}}^{(\mathbb{R})} & \longrightarrow \text{ELLIP}_{R \times L}(2n, j, m_j) \end{array}$$

where

$$\text{ELLIP}_{R \times L}(2n, j, m_j) = \text{ELLIP}_R(2n, j, m_j) \otimes_{(D)} \text{ELLIP}_L(2n, j, m_j),$$

being the global elliptic representation of the bisemisheaf $\theta_{G_{T_R}^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_{T_L}^{(2n)}}^{(\mathbb{R})}$, is the product, right by left, of $2n$ -dimensional real global elliptic semimodules given by:

$$\text{ELLIP}_L(2n, j, m_j) = \bigoplus_{j, m_j} \lambda^{\frac{1}{2}}(2n, j, m_j) e^{2\pi i j x}, \quad x \in (F_{v_1}^+)^n,$$

$$\text{and } \text{ELLIP}_R(2n, j, m_j) = \bigoplus_{j, m_j} \lambda^{\frac{1}{2}}(2n, j, m_j) e^{-2\pi i j x}.$$

Proof. This proposition is an adaptation of proposition 4.2.8. ■

5.1.19 Proposition

- 1) A cuspidal representation, given by the elliptic representation $\text{ELLIP}_{R \times L}(2n, j, m_j)$, corresponds to the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$.

2) On the monodromy bisemisheaves $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{i,\beta_i}$, $1 \leq i \leq k$,
 $1 \leq \beta_i \leq b_i$, above $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$, no cuspidal representation of the elliptic type
 can be found, except if surgeries are performed.

Proof. According to section 5.1.5, the non-singular fibres (or sections) $F_{i\lambda_{jL}}^{(2n-1)}$ (resp. $F_{i\lambda_{jR}}^{(2n-1)}$) of the monodromy sheaf $\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$) are diffeomorphic to the spheres $TS_{iL_j}^{(2n-1)}$ (resp. $TS_{iR_j}^{(2n-1)}$). So, no elliptic representation can be found for the monodromy sheaf $\mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)$ (resp. $\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i)$) since $TS_{iL_j}^{(2n-1)}$ (resp. $TS_{iR_j}^{(2n-1)}$) cannot be transformed bijectively into a $(2n-1)$ -dimensional real semitorus (being able to constitute an equivalence class representative of a global elliptic representation according to proposition 5.1.18). ■

5.1.20 Proposition

On the desingularized bisemisheaf $\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}$ and its monodromy bisemisheaves $\{\mathcal{F}_{F_{i\lambda_R}^{(2n-1)}}(\beta_i) \otimes \mathcal{F}_{F_{i\lambda_L}^{(2n-1)}}(\beta_i)\}_{i,\beta_i}$, the only irreducible global correspondences of Langlands:

$$\begin{array}{ccc} \text{Irr Rep}_{W_{F_R^+ \times L}^{(2n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \longrightarrow & \text{Irr ELLIP}(\theta_{G_{T_R}^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_{T_L}^{(2n)}}^{(\mathbb{R})}) \\ \parallel & & \parallel \\ \theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} & \longrightarrow & \text{ELLIP}_{R \times L}(2n, j, m_j) \end{array}$$

exists.

Proof.

- 1) The above mentioned Langlands correspondence results from the toroidal compactification of the irreducible holomorphic correspondence $\text{Irr hol}^{(2n)}(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})})$, introduced in proposition 5.1.16:

$$\begin{array}{ccc}
\text{Irr Rep}_{W_{F_R \times L}^{(2n)}}(W_{F_R^+}^{ab} \times W_{F_L^+}^{ab}) & \longrightarrow & \text{Irr hol}^{(2n)}(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}) \\
\parallel & \searrow & \parallel \\
\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})} & & f_{\overline{v}}(z^*) \otimes f_v(z) \\
& & \downarrow \tau^{\text{tor}}(\text{Irr hol}_{\theta_{G_R \times L}^{(\mathbb{R})}}^{(2n-1)}) \\
& & \text{Irr ELLIP}(\theta_{G_R^{(2n)}}^{(\mathbb{R})} \otimes \theta_{G_L^{(2n)}}^{(\mathbb{R})}) \\
& & \parallel \\
& & \text{ELLIP}_{R \times L}(2n, j, m_j)
\end{array}$$

- 2) According to proposition 5.1.19, no Langlands correspondence exists for the monodromy bisemisheaves because no bijection can be found between the non-singular fibres, diffeomorphic to TS^{2n-1} , and $(2n-1)$ -dimensional real semitori. ■

5.2 The monodromy of isolated singularities on reducible complex bisemisheaves

5.2.1 Singular reducible bisemisheaves

In chapter 4 of [Pie1], the possible reducibilities of the representation $\text{Rep}(\text{GL}_{2n}(F_{\overline{\omega}} \times F_{\omega}))$ of the bilinear algebraic semigroup $\text{GL}_{2n}(F_{\overline{\omega}} \times F_{\omega})$ were introduced. They are of three types:

- a) partially reducible if:

$$\text{Rep}(\text{GL}_{2n=2n_1+\dots+2n_s}(F_{\overline{\omega}} \times F_{\omega})) = \bigsqcup_{2n_{\ell}=2n_1}^{2n_s} \text{Rep}(\text{GL}_{2n_{\ell}}(F_{\overline{\omega}} \times F_{\omega}))$$

for any partition $2n = 2n_1 + \dots + 2n_{\ell} + \dots + 2n_s$ of $2n$;

- b) orthogonally completely reducible if:

$$\text{Rep}(\text{GL}_{2n=2_1+\dots+2_n}(F_{\overline{\omega}} \times F_{\omega})) = \bigsqcup_{\ell=1}^n \text{Rep}(\text{GL}_{2_{\ell}}(F_{\overline{\omega}} \times F_{\omega}))$$

- c) non orthogonally completely reducible if:

$$\begin{aligned}
\text{Rep}(\text{GL}_{2n_{R \times L}}(F_{\overline{\omega}} \times F_{\omega})) &= \bigsqcup_{2\ell_R=2\ell_L=1}^{2n} \text{Rep}(\text{GL}_{2_{\ell_R \times L}}(F_{\overline{\omega}} \times F_{\omega})) \\
&= \bigsqcup_{2k_R \neq \ell_L} \text{Rep}(T_{2k_R}^t(F_{\overline{\omega}}) \times T_{2\ell_L}(F_{\omega})) .
\end{aligned}$$

The analytic representation spaces over these reducible bilinear algebraic semigroups are respectively the following bisemisheaves of differentiable bifunctions:

- a) the partially reducible bisemisheaf $\theta_{\text{GL}_{2n=2n_1+\dots+2n_\ell+\dots+2n_s}(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ over the bilinear algebraic semigroup $\text{GL}_{2n=2n_1+\dots+2n_\ell+\dots+2n_s}(F_{\overline{\omega}} \times F_{\omega})$;
- b) the orthogonal completely reducible bisemisheaf $\theta_{\text{GL}_{2n=2_1+\dots+2_\ell+\dots+2_r}(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ over the bilinear algebraic semigroup $\text{GL}_{2n=2_1+\dots+2_\ell+\dots+2_r}(F_{\overline{\omega}} \times F_{\omega})$;
- c) the non orthogonal completely reducible bisemisheaf $\theta_{\text{GL}_{2n_{R \times L}}}(F_{\overline{\omega}} \times F_{\omega})^{(\mathbb{C})}$ over the bilinear algebraic semigroup $\text{GL}_{2n_{R \times L}}(F_{\overline{\omega}} \times F_{\omega})$.

Being concerned by the monodromy group action on these (“singular”) bisemisheaves, the only relevant reducible bisemisheaf is the orthogonal completely reducible bisemisheaf. Indeed, the monodromy group action on the partially reducible bisemisheaf decomposing into:

$$\theta_{\text{GL}_{2n=2n_1+\dots+2n_s}(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} = \bigoplus_{n_\ell=n_1}^{n_s} \theta_{\text{GL}_{2n_\ell}(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$$

can amount to the general irreducible case treated in section 5.1 for $n_\ell \geq 2$, $n_1 \leq n_\ell \leq n_s$.

On the other hand, as the non orthogonal completely reducible bisemisheaves

$$\theta_{T_{2^k_R}^t}(F_{\overline{\omega}}) \otimes \theta_{T_{2^\ell_L}^t}(F_{\omega}) \in \theta_{\text{GL}_{2n_{R \times L}}}(F_{\overline{\omega}} \times F_{\omega}) , \quad 1 \leq k, \ell \leq n ,$$

on the off-diagonal algebraic linear semigroups $T_{2^k_R}^t(F_{\overline{\omega}})$ and $T_{2^\ell_L}^t(F_{\omega})$ are generated from the orthogonal ones $\theta_{T_{2^k_R}^t}(F_{\overline{\omega}}) \otimes \theta_{T_{2^\ell_L}^t}(F_{\omega})$, their monodromy group actions are not really pertinent.

So, the only relevant reducible bisemisheaf from the monodromy point of view is the orthogonal completely reducible bisemisheaf

$$\theta_{\text{GL}_{2n=2_1+\dots+2_n}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega}) = \bigoplus_{\ell=1}^n \theta_{\text{GL}_{2_\ell}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$$

whose irreducible elements $\theta_{\text{GL}_{2_\ell}}^{(\mathbb{C})}(F_{\overline{\omega}} \times F_{\omega})$, $1 \leq \ell \leq n$, being able to generate monodromy groups, will be studied in this section.

5.2.2 Critical level sets of the bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$

Let

$$\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} \equiv \theta_{\text{GL}_2(F_{\overline{\omega}})}^{(\mathbb{C})} \otimes \theta_{\text{GL}_2(F_{\omega})}^{(\mathbb{C})} \quad (\equiv \theta_{T_2^t(F_{\overline{\omega}})}^{(\mathbb{C})} \otimes \theta_{T_2(F_{\omega})}^{(\mathbb{C})})$$

be a bisemisheaf of complex-valued differentiable bifunctions $\phi_{G_{j_R}^{(2)}}^{(2)}(z_{g_{j_R}}) \otimes \phi_{G_{j_L}^{(2)}}^{(2)}(z_{g_{j_L}})$, $1 \leq j \leq r \leq \infty$, over the conjugacy class representatives $g_{R \times L}^{(2)}(j, m_j]$ of the complex algebraic bilinear semigroup $G^{(2)}(F_{\overline{\omega}} \times F_{\omega})$.

Assume that, on a domain $U_{j_R}^{(2)} \times U_{j_L}^{(2)} \subset g_{R \times L}^{(2)}[j, m_j]$, each bifunction $\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(z_{g_{j_R}}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(z_{g_{j_L}})$ is locally a Morse bifunction described by:

$$\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)}) = z_{j_1}^2 + z_{j_2}^2, \quad (z_1, z_2) \in \mathbb{C}^2.$$

At $z_{j_1} = z_{j_2} = 0$, the bifunction $\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)})$ has a non-degenerate singularity.

The critical level set of $\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)})$ is the singular fibre $F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)}$ given by:

$$\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)}) = z_{j_1}^2 + z_{j_2}^2 = 0;$$

it consists in two complex lines intersecting at 0 [H-Z].

5.2.3 Proposition

Let

$$F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} = z_{j_1}^2 + z_{j_2}^2 = 0, \quad \forall j, \quad 1 \leq j \leq r \leq \infty,$$

be the singular bifibre of the j -th bisection $\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)})$ of the bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega} \times F_{\omega}})}^{(C)}$.

Then, we have that:

- 1) the corresponding non singular bifibres $F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$ are diffeomorphic to the product, right by left, $T_{\lambda_{j_R}}^2(t) \times T_{\lambda_{j_L}}^2(t)$ of two semitori.
- 2) the homology group $H_1(F_{\lambda_{j_L}}^{(1)}; \mathbb{Z}) \simeq \mathbb{Z}$ (resp. $H_1(F_{\lambda_{j_R}}^{(1)}; \mathbb{Z}) \simeq \mathbb{Z}$) of the semitorus $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) is generated by the upper (resp. lower) semicircle $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$) on $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) in such a way that when its radius tends to the unity, the semicircle shrinks to the singularity and is then called the vanishing semicycle characterized by a rank $r_{\Delta_j^{(1)}} = N$.
- 3) The covanishing semicycle $\nabla_{L_j}^{(1)}$ (resp. $\nabla_{R_j}^{(1)}$) is a line on $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) perpendicular to the vanishing semicycle $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$) and is characterized by a rank $r_{F_{\lambda_j}^{(1)}} \approx j \cdot N$.

Proof.

- 1) The j -th critical level sets are the non-singular bifibres $F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$ described by the equations:

$$z_{j_1}^2(t) + z_{j_2}^2(t) = \lambda_j(t), \quad \lambda \neq 0;$$

they are diffeomorphic to the cylinder $S^1 \times \mathbb{R}^1$.

Indeed, the Riemann surface of the function $z_{j_2}(t) = \sqrt{\lambda_j(t) - z_{j_1}^2}$ is formed from two copies of the complex z_{j_1} -plane glued together along the cut $(-\lambda_j(t), +\lambda_j(t))$ [H-Z]. Each copy of the cut plane is homeomorphic to half a cylinder (and a 2-dimensional semitorus). The line of the cut is a circle on the cylinder encircling the critical value and given by the equation:

$$\lambda_j(t) = r_{\lambda_j(t)} e^{2\pi i j t}, \quad 0 \leq t \leq 1.$$

As t increases, both branch points $z_{j_1} = \pm \sqrt{\lambda_j(t)} = (\pm \sqrt{r_{\lambda_j(t)} e^{2\pi i j t}})$ move around $z_{j_1} = 0$ in the positive direction. As t varies from 0 to 1, each of these points performs a revolution and changes place with the other.

Thus, as $\lambda_j(t)$ encircles the singularity, a corresponding series of pairs $\{T_{\lambda_{j_R}}^2(t), T_{\lambda_{j_L}}^2(t)\}_t$ of two-dimensional semitori are generated.

- 2) The circle on the cylinder encircling the critical value is the vanishing cycle given by the equation:

$$\lambda_j^{(V)}(t) \simeq e^{2\pi i j t},$$

i.e. when the radius $r_{\lambda_j(t)} \simeq 1$.

Indeed, in this case, the left (resp. right) vanishing semicycle $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$), given by the equation:

$$\lambda_j(t) \simeq e^{2\pi i j t},$$

restricted to the upper (resp. lower) half plane, corresponds to a function $\phi_{P_{j_L}}^{(1)}(x_{p_{j_L}})$ (resp. $\phi_{P_{j_R}}^{(1)}(x_{p_{j_R}})$) on the left (resp. right) conjugacy class representative $P^{(2)}(F_{v_1^+}^{+})$ (resp. $P^{(2)}(F_{\bar{v}_1^+}^{+})$) of the linear parabolic subgroup $P^{(2)}(F_{v_1^+})$ (resp. $P^{(2)}(F_{\bar{v}_1^+})$) according to section 5.1.5.

And, thus, the rank of $\Delta_{L_j}^{(1)}$ and of $\Delta_{R_j}^{(1)}$ is given by $r_{\Delta_j^{(1)}} = N$.

- 3) As the covanishing semicycle $\nabla_{L_j}^{(1)}$ (resp. $\nabla_{R_j}^{(1)}$) is a semicircle on $T_{\lambda_{j_L}}^2$ (resp. $T_{\lambda_{j_R}}^2$) perpendicular to $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$), it will be assumed to have a rank $r_{\nabla_{j^{(1)}}} \simeq j \cdot N$ since it is defined on the j -th conjugacy class representative of the linear algebraic semigroup $G^{(2)}(F_\omega)$ (resp. $G^{(2)}(F_{\bar{\omega}})$) (see, for example, proposition 3.2.2. to illustrate this point). ■

5.2.4 Proposition

Let $(\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)}), F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)}, F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t), \Delta_{R_j}^{(1)} \times \Delta_{L_j}^{(1)})$ be the 4-th bituple introduced in proposition 5.2.3.

Then the mapping

$$h_{\gamma_{j_R \times L}}^{(1)} : F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t) \longrightarrow F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$$

of the non singular bifibre into itself is the monodromy of the product, right by left, $\Delta_{R_j}^{(1)} \times \Delta_{L_j}^{(1)}$ of the vanishing semicycles realized by the conjugacy action of the j -th conjugacy class representative of the bilinear algebraic semigroup $G^{(2)}(F_{\bar{v}}^+ \times F_v^+)$ on the corresponding conjugacy class representative of the bilinear parabolic subsemigroup $P^{(2)}(F_{\bar{v}^1} \times F_{v^1})$.

Proof.

- 1) This proposition is a particular case of the one which was treated in proposition 5.1.6 in the sense that the monodromy $h_{\gamma_{j_R \times L}}^{(1)}$ is associated with the injective mapping:

$$I_{\Delta_{\lambda_{j_R \times L}}^{(1)} \rightarrow F_{j_R \times L}^{(1)}} : \Delta_{R_j}^{(1)} \times \Delta_{L_j}^{(1)} \longrightarrow F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$$

inflating the left (resp. right) vanishing semicycle $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$), characterized by a rank $r_{\Delta_j^{(1)}} = N$, into the left (resp. right) non singular fibre $F_{\lambda_{j_L}}^{(1)}(t)$ (resp. $F_{\lambda_{j_R}}^{(1)}(t)$), characterized by a rank $r_{F_{\lambda_j}^{(1)}} = m_j (j \cdot N)$ where m_j is the number of 1D-fibres perpendicular to $\Delta_{L_j}^{(1)}$ (resp. $\Delta_{R_j}^{(1)}$) into $F_{\lambda_{j_L}}^{(1)}(t)$ (resp. $F_{\lambda_{j_R}}^{(1)}(t)$) diffeomorphic to the 2D-semitorus $T_{\lambda_{j_L}}^2(t)$ (resp. $T_{\lambda_{j_R}}^2(t)$).

- 2) The inflation action of $I_{\Delta_{j_R \times L}^{(1)} \rightarrow F_{\lambda_{j_R \times L}}^{(1)}}$ on $\Delta_{R_j}^{(1)} \times \Delta_{L_j}^{(1)}$ corresponds to the conjugation of the j -th conjugacy class representative of $G^{(2)}(F_{\bar{v}}^+ \times F_v^+)$ on the corresponding conjugacy class representative of $P^{(2)}(F_{\bar{v}^1} \times F_{v^1})$. ■

5.2.5 Number of non singular bifibres

Assume that each bisection $\phi_{G_{g_{j_R}}^{(C)}}^{(2)}(U_{j_R}^{(2)}) \otimes \phi_{G_{g_{j_L}}^{(C)}}^{(2)}(U_{j_L}^{(2)})$ of $\theta_{\text{GL}_2(F_{\bar{v}} \times F_v)}^{(C)}$ is endowed with the same singular bifibre $F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} = z_{j_1}^2 + z_{j_2}^2 = 0$.

The number of non singular bifibres $(F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t))$, corresponding to $F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)}$, depends on the expanding phase responsible for the monodromy $h_{\gamma_{j_R \times L}}^{(1)}$ (see section 5.1.2). Indeed, it corresponds to this expanding phase the inverse mapping

$$r_{F_{\lambda_{j_R \times L}}^{(1)} \rightarrow F_{0_{j_R \times L}}^{(1)}}^{-1} : F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} \longrightarrow F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t)$$

of the retraction of the monodromy (see section 5.1.9):

$$r_{F_{\lambda_{j_R \times L}}^{(1)} \rightarrow F_{0_{j_R \times L}}^{(1)}} : F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t) \longrightarrow F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} .$$

Let β be the number of non singular bifibres $(F_{\lambda_{j_R}}^{(1)}(t) \times F_{\lambda_{j_L}}^{(1)}(t))$ above each bisection of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.

5.2.6 Proposition

If each bisection of the bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ is endowed with the same singular bifibre $F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} = z_{j_1}^2 + z_{j_2}^2 = 0$, then a set of β bisemisheaves $\{\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)\}_{b=1}^{\beta}$, isomorphic to the desingularized bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$, can be generated by monodromy if β is the number of non singular bifibres above each bisection of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.

Proof. As there are β identical non singular bifibres $\{F_{\lambda_{j_R}}^{(1)}(b) \times F_{\lambda_{j_L}}^{(1)}(b)\}_{b=1}^{\beta}$ (assuming the one-to-one correspondence $t \leftrightarrow b$) above each bisection of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$, β bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$, $1 \leq b \leq \beta$, can be built from these nonsingular bifibres in such a way that the sections of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$ are in one-to-one correspondence with the sections of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$. Furthermore, the sections of $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$, and the corresponding sections of the monodromy bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$ are isomorphic according to proposition 5.2.4.

So, the monodromy bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$, $1 \leq b \leq \beta$, are isomorphic to (or “copies of”) the original desingularized bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$. \blacksquare

5.2.7 Proposition

Let $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ be a complex bisemisheaf, whose bisections are endowed with the same singular bifibres $F_{0_{j_R}}^{(1)} \times F_{0_{j_L}}^{(1)} = z_{j_1}^2 + z_{j_2}^2 = 0$, $\forall j$, $1 \leq j \leq r \leq \infty$, and let $\{\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)\}_{b=1}^{\beta}$ be the set of β monodromy bisemisheaves.

Then, it results that:

1) a global holomorphic representation:

$$\text{Irr hol}_{\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}}^{(1)} : \theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} \longrightarrow f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m)$$

corresponds to the desingularized ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$;

2) β global holomorphic representations:

$$\text{Irr hol}_{\theta_{G_{R \times L}}^{(\mathbb{C})\text{mon}}(1)} : \theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b) \longrightarrow f_{\overline{\omega}}(z_{m_b}^*) \otimes f_{\omega}(z_{m_b}), \quad 1 \leq b \leq \beta,$$

can be associated with the monodromy bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$.

Proof.

- 1) If $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ is desingularized and if its left (resp. right) linear sections are glued together, a global holomorphic representation can be given to it by the holomorphic bifunction $f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m)$ where:

$$f_{\omega}(z_m) = \sum_{j, m_j} c_{j, m_j} (z_m - z_{m_0})^j$$

$$(\text{resp. } f_{\overline{\omega}}(z_m^*) = \sum_{j, m_j} c_{j, m_j}^* (z_m^* - z_{m_0}^*)^j)$$

with:

- z_m, z_{m_0} (resp. $z_m^*, z_{m_0}^*$) complex (resp. conjugate complex) variables;
- c_{j, m_j} (resp. c_{j, m_j}^*) coefficients (see proposition 5.1.14).

- 2) β global holomorphic representations of monodromy type are given by the holomorphic bifunctions $f_{\overline{\omega}}(z_{m_b}^*) \otimes f_{\omega}(z_{m_b})$, $1 \leq b \leq \beta$, in such a way that they are equivalent to the ground holomorphic bifunction $f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m)$ taking into account the proposition 5.2.6. ■

5.2.8 Proposition

Let $\{\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)\}_{b=1}^{\beta}$ be the β monodromy bisemisheaves above the desingularized ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.

Then, the global holomorphic correspondences are the following:

$$a) \quad \text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) \longrightarrow \text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})})$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} \longrightarrow f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m)$$

where $\text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_R}^{ab} \times W_{F_L}^{ab})$ is the irreducible complex representation of the bilinear global Weil group $(W_{F_R}^{ab} \times W_{F_L}^{ab})$ given by the ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.

$$\begin{array}{ccc}
b) \text{ Irr Rep}_{W_{F_{R \times L}}^{\text{mon}}}^{(1)}(W_{F_{R \text{mon}}}^{ab}(b) \times W_{F_{L \text{mon}}}^{ab}(b)) & \longrightarrow & \text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)) \\
\parallel & & \parallel \\
\theta_{F_{\overline{\omega}} \times F_{\omega}}^{(\mathbb{C})\text{mon}}(b) & \longrightarrow & f_{\overline{\omega}}(z_{m_b}^*) \otimes f_{\omega}(z_{m_b})
\end{array}$$

$1 \leq b \leq \beta$,

for the β monodromy bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$.

Proof. This proposition is an adaptation of proposition 5.1.16 to the products, right by left, of $1D$ -complex (semi)sheaves. \blacksquare

5.2.9 Toroidal compactification

The ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ and the β monodromy bisemisheaves $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)$ are defined over the bilinear algebraic semigroup $\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})$. So, a toroidal compactification of the linear conjugacy class representatives $g_L^{(2)}[j, m_j]$ of $\text{GL}_2(F_{\omega})$ (resp. $g_R^{(2)}[j, m_j]$ of $\text{GL}_2(F_{\overline{\omega}})$) can be realized by the mappings:

$$\begin{aligned}
\tau_{\mathbb{C}}^{\text{tor}}[j, m_j] : \quad & g_L^{(2)}[j, m_j] \longrightarrow g_{T_L}^{(2)}[j, m_j] \\
& c_{j, m_j} z_m^j \longrightarrow \lambda^{\frac{1}{2}}(2, j, m_j) e^{2\pi i j z_m}, \quad z_m \in \mathbb{C}, \\
(\text{resp. } \tau_{\mathbb{C}}^{\text{tor}}[j, m_j] : \quad & g_R^{(2)}[j, m_j] \longrightarrow g_{T_R}^{(2)}[j, m_j] \\
& c_{j, m_j}^* z_m^{*j} \longrightarrow \lambda^{\frac{1}{2}}(2, j, m_j) e^{-2\pi i j z_m}) \quad \forall 1 \leq j \leq r \leq \infty,
\end{aligned}$$

where $g_{T_L}^{(2)}[j, m_j] = \lambda^{\frac{1}{2}}(2, j, m_j) e^{2\pi i j z_m}$ (resp. $g_{T_R}^{(2)}[j, m_j] = \lambda^{\frac{1}{2}}(2, j, m_j) e^{-2\pi i j z_m}$) is a two-dimensional real semitorus localized in the upper (resp. lower) half space.

5.2.10 Proposition

- 1) A cuspidal representation, given right by the product $\text{EIS}_{R \times L}(1, j, m_j) = \text{EIS}_R(1, j, m_j) \times \text{EIS}_L(1, j, m_j)$, of the (truncated) Fourier development of a normalized right cusp form by its left equivalent, can be associated with the desingularized ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$.
- 2) Similarly, a cuspidal representation given by $\text{EIS}_{R \times L}^{\text{mon}}(1, j, m_j)$ corresponds to each monodromy bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}$ on the bilinear algebraic semigroup $\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})$ compactified toroidally.

Proof.

- 1) The toroidal compactification $\tau_{\mathbb{C}}^{\text{tor}}(\text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}))$ of the global holomorphic representation of the ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ generates the corresponding cuspidal representation $\text{Irr cusp}_{\theta_{\text{GL}_2 R \times L}^{(\mathbb{C})}}^{(1)}$ according to:

$$\begin{array}{ccc}
 \text{Irr hol}_{\theta_{\text{GL}_2 R \times L}^{(\mathbb{C})}}^{(1)} : & \theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} & \longrightarrow f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m) \\
 \downarrow \tau_{\mathbb{C}}^{\text{tor}}(\text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})})) & \downarrow & \downarrow \\
 \text{Irr cusp}_{\theta_{\text{GL}_2 R \times L}^{(\mathbb{C})}}^{(1)} : & \theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})} & \longrightarrow \text{EIS}_{R \times L}(1, j, m_j)
 \end{array}$$

where:

- $\text{EIS}_{R \times L}(1, j, m_j) = \text{EIS}_R(1, j, m_j) \times_{(D)} \text{EIS}_L(1, j, m_j)$, being the global cuspidal representation of the ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})}$ is the product, right by left, of the (truncated) Fourier development of the cusp forms [Pie1]:

$$\begin{aligned}
 \text{EIS}_L(1, j, m_j) &= \bigoplus_{j, m_j} \lambda^{\frac{1}{2}}(1, j, m_j) e^{2\pi i j z_m}, \\
 \text{EIS}_R(1, j, m_j) &= \bigoplus_{j, m_j} \lambda^{\frac{1}{2}}(1, j, m_j) e^{-2\pi i j z_m},
 \end{aligned}$$

with $\lambda^{\frac{1}{2}}(1, j, m_j)$ being the square root of the product of the eigenvalues of the coset representative $U_{j, m_{j_R}} \times U_{j, m_{j_L}}$ of the product, right by left, of Hecke operators;

- $\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)$ is the bilinear algebraic semigroup whose conjugacy class representatives $g_{T_R}^{(2)}[j, m_j] \times g_{T_L}^{(2)}[j, m_j]$ have undergone a toroidal compactification.

- 2) Each monodromy bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})\text{mon}}$, having been compactified toroidally, gives rise to a similar cuspidal representation $\text{Irr cusp}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})\text{mon}}) = \text{EIS}_{R \times L}(1, j, m_j)$ since the monodromy bisemisheaves are isomorphic to (or copies of) the ground bisemisheaf according to proposition 5.2.6. ■

5.2.11 Proposition

On the desingularized ground bisemisheaf $\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}$ and its monodromy bisemisheaves $\{\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)\}_{b=1}^{\beta}$, we have the following irreducible Langlands global correspondences:

$$\begin{array}{ccc}
a) \quad \text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) & \longrightarrow & \text{Irr cusp}(\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})}) \\
\parallel & & \parallel \\
\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} & \longrightarrow & \text{EIS}_{R \times L}(1, j, m_j) \\
b) \quad \text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_{R\text{mon}}}^{ab}(b) \times W_{F_{L\text{mon}}}^{ab}(b)) & \longrightarrow & \text{Irr cusp}(\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})\text{mon}}(b)) \\
\parallel & & \parallel \\
\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b) & \longrightarrow & \text{EIS}_{R \times L}^{\text{mon}}(1, j, m_j)_b
\end{array}$$

Proof. These Langlands global correspondences result from the preceding sections summarized in the two following diagrams:

$$\begin{array}{ccccc}
a) \quad \text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_R}^{ab} \times W_{F_L}^{ab}) & \longrightarrow & \text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})}) & \xrightarrow{\tau_{\mathbb{C}}^{\text{tor}}} & \text{Irr cusp}(\theta_{\text{GL}_2(F_{\overline{\omega}}^T \times F_{\omega}^T)}^{(\mathbb{C})}) \\
\parallel & & \parallel & & \parallel \\
\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})} & \longrightarrow & f_{\overline{\omega}}(z_m^*) \otimes f_{\omega}(z_m) & \xrightarrow{\tau_{\mathbb{C}}^{\text{tor}}} & \text{EIS}_{R \times L}(1, j, m_j)_b \\
b) \quad \text{Irr Rep}_{W_{F_R \times L}}^{(1)}(W_{F_{R\text{mon}}}^{ab}(b) \times W_{F_{L\text{mon}}}^{ab}(b)) & \longrightarrow & \text{Irr hol}^{(1)}(\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b)) & \xrightarrow{\tau_{\mathbb{C}}^{\text{tor}}} & \text{Irr cusp}(\theta_{\text{GL}_2(F_{\overline{\omega}}^+ \times F_{\omega}^+)}^{(\mathbb{C})\text{mon}}(b)) \\
\parallel & & \parallel & & \parallel \\
\theta_{\text{GL}_2(F_{\overline{\omega}} \times F_{\omega})}^{(\mathbb{C})\text{mon}}(b) & \longrightarrow & f_{\overline{\omega}}(z_{m_b}^*) \otimes f_{\omega}(z_{m_b}) & \xrightarrow{\tau_{\mathbb{C}}^{\text{tor}}} & \text{EIS}_{R \times L}^{\text{mon}}(1, j, m_j)_b
\end{array}$$

■

References

- [Abh] ABHYANKAR, S.: Resolution of singularities and modular Galois theory. *Bull. Amer. Math. Soc.*, **38** (2000), 131–169.
- [Abi] ABIKOFF, W.: The residual limit sets of Kleinian groups. *Acta Math.*, **130** (1973), 127–144.
- [A-G-L-V] ARNOLD, V., GORYUNOV, V., LYASHKO, O., AND VASIL’EV, V.: Singularity theory I. *Springer* (1998).
- [Ahl] AHLFORS, L.: Fundamental polyhedrons and limit point sets of Kleinian groups. *Proc. Nat. Acad. Sci. (N.Y.)*, **55** (1966), 251–254.
- [Arn1] ARNOLD, V.I.: Lectures on bifurcations in versal families. *Russ. Math. Surv.*, **27** (1972), 54–123.
- [A-V-G1] ARNOLD, V., VARCHENKO, A., AND GOUSSEIN-ZADE, S.: Singularités des applications différentiables. *Mir*. (1982).
- [Ber] BERTHELOT, P.: Altérations de variétés algébriques. *Sém. Bourbaki*, **815** (1995-96).
- [Ber1] BERKOVICH, V.: Vanishing cycles for formal schemes, I. *Invent. Math.*, **115** (1994), 539–571.
- [Ber2] BERKOVICH, V.: Vanishing cycles for formal schemes, II. *Invent. Math.*, **125** (1996), 367–390.
- [B-L-M-P] BAMON, R. LABARCA, R., MANE, R. AND PACIFICO, M.J.: The explosion of singular cycles. *Publ. Math. I.H.E.S.*, **78** (1993), 207–232.
- [Cam1] A’CAMPO, N.: Le groupe de monodromie du déplacement des singularités isolées de courbes planes. *Math. Ann.*, **213** (1975), 1–32.
- [Cam2] A’CAMPO, N.: Monodromy of real isolated singularities. *ArXiv Math.*, AG–0301006.
- [Car] CARTAN, E.: Leçons sur la géométrie des espaces de Riemann. Éd. J. Gabay, *Gauthier-Villars*. (1988).
- [Chm] CHMUTOV, S.V.: Monodromy group of critical points of functions. *Invent. Math.*, **67** (1982).
- [DeJ] DE JONG, A.J.: Smoothness, semistability and alterations. *Publ. Math. I.H.E.S.*, **83** (1996), 51–93.
- [Del1] DELIGNE, P.: Le formalisme des cycles évanescents. SGA7, Exposé XIII, *Lect. Notes Math.*, **340** (1973), Springer.

- [Del2] DELIGNE, P.: Comparaison avec la théorie transcendante. SGA7, Exposé XIV, *Lect. Notes Math.*, **340** (1973), Springer.
- [Del3] DELIGNE, P.: La formule de Picard-Lefschetz. *Lect. Notes Math.*, **340** (1973), 165–196, Springer.
- [Ebe1] EBELING, W.: The monodromy group of isolated singularities of complete intersections. *Lect. Notes Math.*, **1293**, Springer.
- [Ebe2] EBELING, W.: An arithmetic characterisation of the symmetric monodromy group of singularities. *Invent. Math.*, **77** (1984), 85–99.
- [E-R] ECKMAN, J.P. AND RUELLE, D.: Ergodic theory of chaos and strange attractors. *Rev. Mod. Phys.*, **57** (1985), 617–655.
- [G-K] GRAUERT, H., AND KERNER, H.: Deformationen von Singularitäten komplexer Räume. *Math. Ann.*, **153** (1964), 236–260.
- [G-R] GRAUERT, H., AND REMMERT, R.: Coherent analytic sheaves. *Ser. Compr. Stud. in Math.*, **265** (1984), Springer.
- [Gri] GRIFFITHS, P.: Periods of integrals on algebraic manifolds. *Bull. Amer. Math. Soc.*, **76** (1970), 228–296.
- [Gro] GROTHENDIECK, A.: Résumé des premiers exposés de A. Grothendieck, rédigés par P. Deligne. *Lect. Notes Math.*, **288** (1972), 1–25.
- [Hau] HAUSER, H.: The Hironaka theorem on resolution of singularities. *Bull. Amer. Math. Soc.*, **40** (2003), 323–403.
- [Hir1] HIRONAKA, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero. *Annals of Math.*, **79** (1964), 109–326.
- [Hir2] HIRONAKA, H.: Desingularization of excellent surfaces. *Lect. Not. Math.*, **1101** (1984), 99–132.
- [Hir3] HIRONAKA, H.: On resolution of singularities (characteristic zero). *Proceed. Intern. Congres Math.* (1962), 507–521.
- [H-Z] HUSEIN-ZADE, J.M.: The monodromy groups of isolated singularities of hypersurfaces. *Russ. Math. Surv.*, **32** (1977), 23–69.
- [Lan] LANGLANDS, R.: Automorphic representations, Shimura varieties and motives. *Proc. Symp. Pure Math.*, **33** (1977).
- [Lev] LEVINE, R.: Singularities of differentiable mappings. In: *Proceed. of Liverpool singularities I* (Lect. Notes Math. Vol. 192, pp. 1–89). Berlin, Heidelberg, New York: Springer 1971.

- [Mal] MALGRANGE, B.: Le théorème de préparation en géométrie différentiable. In: Topologie différentielle, Secrét. Math., 11, rue Pierre Curie, Paris, 1964 (Sém. H. Cartan **11** (1962–63)).
- [Mat1] MATHER, J.: Stability of C^∞ mappings. I. The division theorem. *Annals of Math.*, **87** (1968), 89–104.
- [Mat2] MATHER, J.: Stability of C^∞ mappings. II. Infinitesimal stability implies stability. *Annals of Math.*, **89** (1969), 254–291.
- [Mat3] MATHER, J.: Stability of C^∞ mappings. III. Finitely determined map germs. *Publ. Math. I.H.E.S.*, **35** (1968), 127–156.
- [Mil1] MILNOR, J.: Hyperbolic geometry, the first 150 years. *Bull. Amer. Math. Soc.*, **6** (1982), 9–24.
- [Mil2] MILNOR, J.: On the concept of attractor. *Com. Math. Phys.*, **99** (1985), 177–195.
- [M-P] MORALES, C.A., AND PUJALS, E.: Singular strange attractors on the boundary of Morse-Smale systems. *Ann. Scient. Éc. Norm. Sup.*, **30** (1997), 693–717.
- [Mum] MUMFORD, D.: The red book of varieties and schemes. *Lect. Not. Math.*, **1358** (1988), Springer.
- [Pel] PELLIKAAN, R.: Finite determinacy of functions with non-isolated singularities. *Proc. London Math. Soc.*, **57** (1988), 357–382.
- [Pie1] PIERRE, C.: n -dimensional global correspondences of Langlands. *ArXiv Math.*, RT/0510348 (2005).
- [Pie2] PIERRE, C.: The geometry of the versal unfolding. Preprint (1992).
- [Pie3] PIERRE, C.: Versal unfolding and strange attractors. Preprint (1992).
- [Pie4] PIERRE, C.: Algebraic quantum theory. Preprint *ArXiv math-ph*/0404024 (2004).
- [P-R] PACIFIO, M.J., ROVELLA, A.: Unfolding contracting singular cycles. *Ann. Scient. Éc. Norm. Sup.*, **26** (1993), 691–700.
- [Rue1] RUELLE, D.: Strange attractors. *Math. Intell.*, **2** (1980), 126–140.
- [Rue2] RUELLE, D.: Small random perturbations and the definition of attractors. *Lect. Notes Math.*, **1007** (1983), 663–676.
- [Sch] SCHUSTER, H.: Deterministic chaos. V.C.M. Editions (1988).
- [Ser1] SERRE, J.-P.: Faisceaux algébriques cohérents. *Annals of Math.*, **61** (1955), 197–278.

- [Sie] SIERMA, D.: Isolated line singularities. In: Singularities Arcana 1981, part 2 (Proc. of symposium in pure math., Vol. 40, pp. 485–496), Amer. Math. Soc. 1982.
- [Sma] SMALE, S.: Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, **73** (1967), 747–817.
- [Ste] STEVENS, J.: Deformations of singularities. *Lect. Not. Math.*, **1811** (2003), Springer.
- [Tak] TAKENS, F.: Partially hyperbolic fixed points. *Topology*, **10** (1971), 133–147.
- [Tho1] THOM, R.: Les singularités des applications différentiables. *Ann. Inst. Fourier*, **6** (1955), 43–87.
- [Tho2] THOM, R.: Stabilité structurelle et morphogénèse. *Interéditions*, Paris (1977).
- [Tou] TOUGERON, J.C.: Idéaux de fonctions différentiables. *Ann. Inst. Fourier*, **18** (1968), 177–240.
- [Tuk1] TUKIA, P.: An isomorphism of geometrically finite Möbius group. *Publ. Math. I.H.E.S.*, **61** (1985), 177–214.
- [Tuk2] TUKIA, P.: The Hausdorff dimension of the limit set of a geometrically finite Kleinian group. *Acta Math.*, **152** (1984), 127–140.
- [Wil] WILLIAMS, R.F.: Expanding attractors. *Publ. Math. I.H.E.S.*, **43** (1974), 169–203.
- [Zar1] ZARISKI, O.: The reduction of the singularities of an algebraic surface. *Annals of Math.* **40** (1939), 639–689.
- [Zar2] ZARISKI, O.: Reduction of the singularities of algebraic three dimensional varieties. *Annals of Math.* **45** (1944), 472–542.
- [Zar3] ZARISKI, O.: Analytical irreducibility of normal varieties. *Annals of Math.* **49** (1948), 352–??.